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35246 U. S. 19 North Suite 215, Palm Harbor, FL 34684, U.S.A.

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A.K. ARINGAZIN Department of Theoretical Physics, Institute for Basic Research, Eurasian National University, Astana 010008, Kazakhstan

aringazin@mail.kz

P.K. BANERJI Department of Theoretical Physics, Faculty of Science, I.N.V. University, Jodhpur-342 2005, India

bannerjpk@yahoo.com

A. BAYOUMI Alazhar University Faculty of Science Mathematics Department, Naser City 11884 Cairo Egypt

draboubakr@yahoo.com
QING-MING CHENG Department of Mathematics, Faculty of Science Josai University, Sakado, Saitama 350-02 Japan

cheng@math.josai.ac.jp
A. HARMANCY Department of Mathematics, University of Hacettepe, Beytepe Campus, Ankara, Turkey

harmaci@hacettepe.edu.tr
L. P. HORWITZ Department of Physics, Tel Aviv University, Ramat Aviv, Israel

horwitz@taunivm.tau.ac.il
C. P. JOHNSON Department of Mathematics, Mississippi State University P. O. Box MA, Mississippi State, MS 39762

cjohnson@math.msstate.edu
S. L. KALLA Department of Mathematics, Vyas Institutes of Higher Education, Jodhpur 342008 India

shyamkalla@gmail.com

N. KAMIYA University of Aizu Center for Mathematical Sciences, Tsuruga, Ikki-machi Aizu-Wakamatsu City Fukushima, 965-8580 Japan

L. LOHMUS* Estonia Academy of Sciences, 142 Rita Street, Tartu 202400, Estonia

R. MIRON Faculty of Mathematics University "Al. I. Cuza" 6600 Isai, Romania rmiron@uaic.ro

M.R. MOLAEI Department of Mathematics, University of Kerman, P.O. Box 76135-133 Kerman, Iran

mrmolaei@mail1.uk.ac.ir
P. NOWOSAD Mathematics FFCLRP - USP, Univ. of San Paulo 14040-901 Ribeirao Preto SP, Brazil

KAR-PING SHUM Institute of Mathematics, Yunnan University, Kunming, China

kpshum@ynu.edu.cn
S. SILVESTROV School of Education, Culture and Communication (UKK) Mälardalen University, Box 883, 71610 Västerås, Sweden

sergei.silvestrov@mdh.se
H. M. SRIVASTAVA Department of Mathematics and Statistics, University of Victoria, Victoria, B.C. V8W 3P4, Canada

hmsri@uvvm.uvic.ca
G. F. TSAGAS* Mathematics Department, Aristotle University Thessaloniki 54006, Greece

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Center for Mathematical Sciences
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Departamento de Física, ESFM, Instituto Politécnico Nacional
Col. Lindavista CP 07738 CDMX, México

J. López-Bonilla, S. Vidal-Beltrán

ESIME-Zacatenco, Instituto Politécnico Nacional
Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

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Department of Mathematics
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6600 Iasi, Romania

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ON GEOMETRIC CONSTRUCTIONS OF FINITE
GROUPS AND LIE ALGEBRAS

Igor Bayak

Belarus, 230025, Grodno Av.
Kosmonavtov 4V 45
bayak@tut.by

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Abstract

In this paper we investigate the topological connection between Abelian and non-Abelian parity groups. Abelian parity groups are formed as kernels of parity homomorphisms in the group $\mathbb{Z}^n$ and non-Abelian parity groups are formed as kernels parity homomorphisms in the group $S_2 \wr S_n$. Factorization of nodes using the Abelian parity group, we obtain a quotient lattice that serves as a one-dimensional cell complex (framework) of the corresponding product of spheres. It is shown that the automorphisms of this quotient lattice form the corresponding non-Abelian parity group. The paper also presents geometric constructions of Lie algebras, in particular $sl_n(\mathbb{C})$, $su(n)$, which are implemented as torus rotations.

MSC: 20B05

Keywords: group of monomial permutations, integer lattice, factorization of integer lattice, automorphisms of bundles of circles, lie algebras of tori

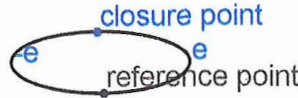
1 Introduction

In this paper, using geometric construction of bundles of circles, we establish a correspondence between finite Abelian and non-Abelian groups. And since this

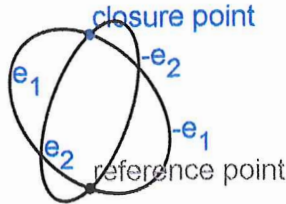


question has not yet been reflected in the literature, for an initial acquaintance with the topic, we suggest turning to examples of automorphism groups of small-dimensional bundles of circles. consisting of discrete topological maps of bundles on themselves, which are obtained as a result of homotopy, allowing continuous change in the radii and location of the circles of the bundle, but prohibiting their intersections and breaks.

First of all, we note that the automorphism group of a bouquet of two halves of circle folded into an eight can be represented by a multiplicative group of two elements $\{1, -1\}$, and the group of automorphisms of a circle divided into two halves of a circle by a closure point consists of a single element $\{1\}$, since without breaking the circle at the closure point, only an identical mapping of the circle to itself is possible. In turn, the automorphism group of a bundle of two circles with

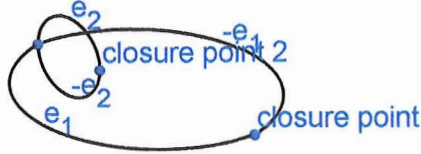


one closure point of the four halves of two circles can be represented by a mul-

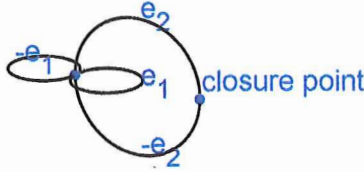


tiplicative matrix group $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\},$

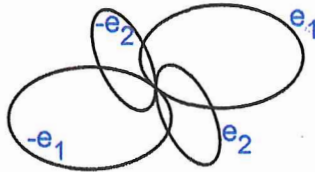
and the automorphism group of a bundle of two circles with two closure points of two pairs of halves of circles can be represented by a multiplica-



tive matrix group $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}$. Simi-



larly, a bundle of eight and a circle generates a group $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ and, finally, a bundle of two circles folded into eights generates a group $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$.



We now use Abelian groups to construct the bundles of circles presented here. Suppose we have a one-dimensional lattice $\mathbb{L}$, which is a straight line with integer nodes, that is, $\mathbb{L} = \{\mathbb{Z}; \mathbb{R}\}$. Then a bunch of circles in the form

of an eight is obtained as a result of factorization $\{\mathbb{Z}/\mathbb{Z}; \mathbb{R}/2\mathbb{Z}\}$, and a bunch in the form of a circle is obtained as a result of factorization $\{\mathbb{Z}/2\mathbb{Z}; \mathbb{R}/2\mathbb{Z}\}$. In turn, if we have a two-dimensional lattice $\mathbb{L}^2 = \{\mathbb{Z}^2; (\mathbb{Z}, \mathbb{R}), (\mathbb{R}, \mathbb{Z})\}$ and its axial constraint $\mathbb{I}^2 = \{(0, \mathbb{Z}), (\mathbb{Z}, 0); (\mathbb{Z}, \mathbb{R}), (\mathbb{R}, \mathbb{Z})\}$, then the lattice factorization $\{\mathbb{Z}^2/*; (\mathbb{Z}, \mathbb{R}/2\mathbb{Z}), (\mathbb{R}/2\mathbb{Z}, \mathbb{Z})\}$ collapses the lattice axes into the corresponding bundle of circles. Indeed, if, as a subgroup (*) of factorizing the lattice nodes, we take a subgroup consisting of pairs of even and odd integers, then we collapse the lattice axes into a bundle of two circles with one closure point, and if we take a subgroup of pairs of even integers, we get a bundle of two circles with two closure points. If $* = \mathbb{Z}^2$, then the lattice axes are compactified into a bundle of a pair of eights.

So, we have clearly shown that in the two-dimensional case there is a topological connection between Abelian and non-Abelian groups, and now it remains only to formalize this result for the case of n dimensions. However, the question of the one-to-one correspondence between Abelian and non-Abelian finite groups remains outside the scope of our study, but it is worth noting that if this assumption is true, then the question of the classification of finite groups becomes trivial.

2 Parity homomorphisms

It is well known that permutation group S_n allows expansion to the group of monomial permutations $P_n = S_2 \wr S_n$, which can be easily represented as a group of such linear transformations $\mathbb{R}^n \rightarrow \mathbb{R}^n : (x_1, \dots, x_n) \rightarrow (x'_1, \dots, x'_n) : x'_i = \pm x_j$, in which the mapping of the set of coordinate indices is bijective, or by a group of square matrices of order n , having in each column and in each row one non-zero element equal to 1 or -1 .

However, it is not well known that three types can be set on the P_n group parity functions. Indeed, by definition $S_2 \wr S_n = \prod^n S_2 \rtimes S_n$, where the group S_n acts on the group $\prod^n S_2$ by permutations of the components of the direct product. Therefore, every element $z \in S_2 \wr S_n$ decomposes into the product $z = xy$, where $x \in S_n$ and $y = (y_1, \dots, y_n) \in \prod^n S_2$, and its conjugate element $z^* \in S_2 \wr S_n$ decomposes into a product $z^* = yx$. Then you can give the following parity definitions of the z element and z^* its conjugate element.

2.1 Definition. The parity function of the first type is called a function $\text{sgn } z = \text{sgn } z^* = \text{sgn } x$

2.2 Definition. The parity function of the second type is called a function $\text{sgn } z = \text{sgn } z^* = \text{sgn } y = \text{sgn } y_1 * \dots * \text{sgn } y_n$

2.3 Definition. The parity function of the third type is called a function $\text{sgn } z = \text{sgn } z^* = \text{sgn } x * \text{sgn } y$

All these functions homomorphically are mapped into a group $\{\pm 1\}$. Indeed, let be given a decomposition of $x = x'x''$ and $y = y'y''$. Then, if $\text{sgn } xy = \text{sgn } x$, then $\text{sgn } x'y' * \text{sgn } x''y'' = \text{sgn } x' * \text{sgn } x'' = \text{sgn } x'x'' = \text{sgn } x = \text{sgn } xy$, if $\text{sgn } xy = \text{sgn } y$, then $\text{sgn } x'y' * \text{sgn } x''y'' = \text{sgn } y' * \text{sgn } y'' = \text{sgn } y'y'' = \text{sgn } y = \text{sgn } xy$, if $\text{sgn } xy = \text{sgn } x * \text{sgn } y$, then $\text{sgn } x'y' * \text{sgn } x''y'' = \text{sgn } x' * \text{sgn } y' * \text{sgn } x'' * \text{sgn } y'' = \text{sgn } x' * \text{sgn } x'' * \text{sgn } y' * \text{sgn } y'' = \text{sgn } x * \text{sgn } y = \text{sgn } xy$, which proves that all our parity functions are homomorphisms. Thus, parity homomorphisms distinguish three subgroups in the P_n group: AP_n , BP_n , CP_n , which are formed as kernels of the parity functions of the first, second, and third types, respectively.

We now examine the algebraic structure of these groups. First of all recall that the group P_n is isomorphic to the group $\mathbb{Z}_2^n \rtimes S_n$, where $\mathbb{Z}_2^n$ this is a direct product of the n component two-element field $\mathbb{Z}_2$, and note that the homomorphism parity of the group $\mathbb{Z}_2^n$, equal to the sum modulo 2 all n component of its element, selects a subgroup in it $A\mathbb{Z}_2^n$, consisting of elements in which 1 occurs an even number of times or does not occur at all. Because every element of the group $A\mathbb{Z}_2^n$ is decomposed into the sum, each term of which consists of pairs of units and the rest zeros, then the group $A\mathbb{Z}_2^n$ is generated by its subgroups isomorphic to $A\mathbb{Z}_2^2$. Then from definitions of groups AP_n , BP_n it follows that the group AP_n is isomorphic to the group $\mathbb{Z}_2^n \rtimes A_n$, where A_n is an alternating group, and the group BP_n is isomorphic group $A\mathbb{Z}_2^n \rtimes S_n$. Because an alternating group is generated by cycles of length 3, then the group AP_n is generated by its third-degree subgroups AP_3 . In turn, since S_n is generated by transpositions a the group $A\mathbb{Z}_2^n$ is generated by its two-component subgroups, then the group BP_n is generated by its subgroups second degree BP_2 . Let's write out a linear representation here and matrix image of the group BP_2 :

$$\begin{aligned} \{(x_1, x_2) \mapsto (x_1, x_2), (x_1, x_2) \mapsto (x_2, x_1), \\ (x_1, x_2) \mapsto (-x_1, -x_2), (x_1, x_2) \mapsto (-x_2, -x_1)\}, \\ \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\} \end{aligned} \quad (2.1)$$

The elements of this group correspond to the reflections of the plane (x_1, x_2) relative to the diagonals $x_2 = x_1$ and $x_2 = -x_1$. We will also write out a linear

representation and a matrix image of the group CP_2 :

$$\begin{aligned} \{(x_1, x_2) \mapsto (x_1, x_2), (x_1, x_2) \mapsto (x_2, -x_1), \\ (x_1, x_2) \mapsto (-x_1, -x_2), (x_1, x_2) \mapsto (-x_2, x_1)\}, \\ \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \end{aligned} \quad (2.2)$$

The elements of this group correspond to the rotations of the Euclidean plane by an angle multiple of $\pi/2$. The CP_n group is also generated by its own subgroups that are isomorphic to CP_2 . Indeed, take an arbitrary element of the group CP_n and multiplying it by elements groups that convert only a pair of space coordinates first, we get an identical map in component S_n and then in the component S_2 of the woven products, which is possible due to the property of the groups S_n and AZ_2^n . Then, the product of inverse elements the expansion will be equal to the original element CP_n , and therefore, any element of the CP_n group can be decomposed into product of elements of groups isomorphic to CP_2 .

Let now be given such a partition of the set $I = \{1, \dots, n\}$ by disjoint subsets of $I_1, \dots, I_m$, which means subsets of I_i are equal to n_i , with $n_1 + \dots + n_m = n$. Then any partition $J = \{I_1, \dots, I_m\}$ can be map a group formed by an external semi-direct product

$$JP_n = CP_{n_1} \times \dots \times CP_{n_m} \rtimes BP_m, \quad (2.3)$$

where the group BP_m acts on the group $\prod^m CP_{n_i}$ corresponding permutations of the direct product components. Let's call this group a *finite non-Abelian parity group*. Note that in General, the JP_n group does not have matrix representation, but its elements can be represented matrices (representing the group of BP_m), the non-zero elements which are groups CP_{n_i} that also have a matrix representation.

Let now be given a matrix image of a real linear representations of second-degree groups CP_2 , BP_2 . Denote using C and B matrix algebras generated by matrices, the corresponding groups, and by C^* , B^* we denote the group of invertible elements in the respective algebras. Then it's easy show that the kernel of the parity homomorphism is $C^* \rightarrow \mathbb{R}^* : C^* \rightarrow \det C^*$ equals special orthogonal group $SO(2)$, and the kernel of the parity homomorphism $B^* \rightarrow \mathbb{R}^* : B^* \rightarrow \det B^*$ equals the special pseudo-orthogonal group $SO(1, 1)$. Indeed, since $C = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \right\}$, where $x, y \in \mathbb{R}$, then the set of solutions to the equation $\det^* = x^2 + y^2 = 1$ just allocates the subgroup $SO(2)$ in the group C^* . In turn, since

$B = \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} \right\}$, where $x, y \in \mathbb{R}$, then the set of solutions to the equation $\det B^* = x^2 - y^2 = 1$ allocates the subgroup $SO(1, 1)$ in the group B^* . Using block-diagonal matrices of the order n , we will set an isomorphic $SO(2)$ group $SO_{jk}(2) = \text{diag} [1, \dots, SO(2)_{(jk)}, \dots, 1]_n$, which is formed by matrices that differ from the single one only in that at the intersection of a pair of rows and a pair of columns with indexes j, k there is a matrix element of the group $SO(2)$. Similarly, set the isomorphic $SO(1, 1)$ group $SO_{jk}(1, 1) = \text{diag} [1, \dots, SO(1, 1)_{(jk)}, \dots, 1]_n$. Then any partition of J can be mapped to a group

$$SO(n_1, \dots, n_m) = \langle SO_{jk}(2), SO_{jk}(1, 1) \rangle_J, \quad (2.4)$$

generated by the generators $SO_{jk}(2)$ and $SO_{jk}(1, 1)$ so that the index pair of the first generator belongs to an arbitrary one a subset of I_i , and a pair of indexes of the second generator belongs to an arbitrary pair of subsets in the partition. However note that sufficient to form a group $SO(n_1, \dots, n_m)$ the number of generators is given by the formula

$$p = \sum_m n_i(n_i - 1)/2 + m(m - 1)/2, \quad (2.5)$$

since for a hyperbolic rotation between two subspaces, only one generator $SO_{jk}(1, 1)$ with arbitrary indexes j, k taken from two different subsets of I_i is sufficient, and the remaining generators $SO_{jk}(1, 1)$ (with other indexes) will be derived from the selected generator and discrete rotations within each of the two subspaces separately. Thus, since the Lie group is generated by its one-parameter subgroups, we get p - the parametric Lie group $SO(n_1, \dots, n_m)$, which let's call a *Lie parity group*.

So, taking the group $S_2 \wr S_n$ as a basis and setting parity homomorphisms on it, we managed to get not only finite but also continuous parity groups, including some classical groups. Note, however, that in order to form all possible matrix Lie groups by generating them with small-dimensional Lie subgroups, it is necessary to use another matrix algebra, namely, the algebra $A = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\}$, where $x, y \in \mathbb{R}$. Then, every subgroup of the General linear group $GL(n, \mathbb{R})$ is generated by all possible small-dimensional Lie groups that can be formed from matrix algebras $M(2, \mathbb{R})$, A, B, C and extended by a group of monomial substitutions of a two-element basis. For example, multi-connected, or rather $2^n n!$ - component, a Lie group consisting of n - matrices, each row and each column of which has one non-zero element, is generated by generators that are isomorphic to a single-connected

Lie group $A^+ = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\}$, where $x, y \in \mathbb{R}^+$, and generators isomorphic to the finite group $S_2 \wr S_2$.

Let us now proceed to the construction of Abelian parity groups that are formed as nuclei of parity homomorphisms in the group $\mathbb{Z}^n$. To do this, set the parity function on the group $\mathbb{Z}^n \rightarrow \mathbb{Z}_2 : |x_1 + \dots + x_n| \bmod 2$, whose value is equal to the sum modulo 2 of all n components of its element. Thus, we get a homomorphism of the group $\mathbb{Z}^n$ to the group $\mathbb{Z}_2$. The kernel of the homomorphism we denote $A\mathbb{Z}^n$ and call it an even group elements $\mathbb{Z}^n$. It is easy to notice that the group $A\mathbb{Z}^n$ consists of elements $\mathbb{Z}^n$ in which odd components occur an even number of times or not at all meet, and therefore it is generated by all sorts of their subgroups of the second degree $A\mathbb{Z}^2$. Meaning that $A\mathbb{Z}^1 = 2\mathbb{Z}$, form also in $\mathbb{Z}^n$ subgroup $B\mathbb{Z}^n = (2\mathbb{Z})^n$, consisting of a direct product (sum) of n instances $A\mathbb{Z}^1$, i.e. from even integers in all components $\mathbb{Z}^n$. Note, however, that $\mathbb{Z}^n/B\mathbb{Z}^n = \mathbb{Z}_2^n$.

Let us then have the partition $J = \{I_1, \dots, I_m\}$, defined earlier. Then, according to this partition, you can form a group

$$J\mathbb{Z}^n = A\mathbb{Z}^{n_1} \times \dots \times A\mathbb{Z}^{n_m}. \quad (2.6)$$

The quotient group $\mathbb{Z}^n/J\mathbb{Z}^n$ isomorphic to the group $\mathbb{Z}_2^m$ of order 2^m we will call a *finite Abelian parity group*. The order of the group $\mathbb{Z}^n/J\mathbb{Z}^n$ can also be calculated based on the fact that

$$\mathbb{Z}^n/J\mathbb{Z}^n = \mathbb{Z}^{n_1}/A\mathbb{Z}^{n_1} \times \dots \times \mathbb{Z}^{n_m}/A\mathbb{Z}^{n_m} \quad (2.7)$$

and every group $\mathbb{Z}^{n_i}/A\mathbb{Z}^{n_i}$ consists of two elements. As an illustration, here are a few examples. If $J = \{(1), (2)\}$, then $J\mathbb{Z}^2 = \{2\mathbb{Z}, 2\mathbb{Z}\}$, and the group $\mathbb{Z}^n/J\mathbb{Z}^2$ consists of four points. If $J = \{(1, 2)\}$, then $J\mathbb{Z}^2 = \{(2\mathbb{Z}, 2\mathbb{Z}), (2\mathbb{Z} + 1, 2\mathbb{Z} + 1)\}$, and the group $\mathbb{Z}^n/J\mathbb{Z}^2$ consists of two points, and on its planes can be represented as two classes of integer parallelograms, i.e. parallelograms made up of pairs of integers on its sides, with vertices equidistant from the zero point lying in even and odd points of the coordinate axes, respectively.

3 Topological connection between parity groups

Let $\mathbb{L} = \{\mathbb{Z}; \mathbb{R}\}$ be a one-dimensional integer lattice in which $z \in \mathbb{Z}$ are lattice nodes, $[2z, 2z + 1]$ are even edges, and $[2z + 1, 2z]$ are odd edges. We now identify

the nodes of our lattice using factorization by the groups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/\mathbb{Z}$, respectively, and identify the edges using the parity equivalence relation. Then, in the first case, we get compactification one-dimensional lattice $\mathbb{L}/2\mathbb{Z}$ as a circle with two factor nodes lying at opposite points on the circle that divide the odd and even factor-ribs, and in the second case, we get compactification one-dimensional lattice $\mathbb{L}/\mathbb{Z}$ in the form of two touching circles with one factor node at the tangent point of circles and a pair of (even and odd) factor-ribs rolled into a circle.

If we fix the zero point (even quotient node) of the quotient lattice $\mathbb{L}/2\mathbb{Z}$ lying on the plane, then the isomorphisms of such an embedding of the lattice on the plane are reduced to a single identical transformation, since without breaking the lattice in an odd quotient node, it is impossible to swap even and odd quotient edges. If we fix the zero point of the quotient lattice $\mathbb{L}/\mathbb{Z}$ lying on the plane, we get an isomorphism of the identical lattice transformation and an isomorphism that swaps even and odd edges, since now there are no topological obstacles to such a permutation.

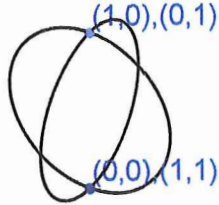
Let in the space $\mathbb{R}^n$ be given an integer lattice

$$\mathbb{L}^n = \{\mathbb{Z}^n; (\mathbb{Z}, \dots, \mathbb{R}, \dots, \mathbb{Z})\}$$

consisting of straight lines intersecting at points $\mathbb{R}^n$ with integer coordinates and parallel basic orts. An integer lattice $\mathbb{L}^n$ is a regularly repeated set of nodes and edges that can be obtained by infinite repetition of a n - dimensional forming parallelepiped consisting of 2^n nodes connected by edges. Hence, the group of isomorphisms of an integer lattice coincides with the group of isomorphisms of the forming parallelepiped, and is therefore equal to the group of monomial substitutions, which is isomorphic to the woven product $S_2 \wr S_n$.

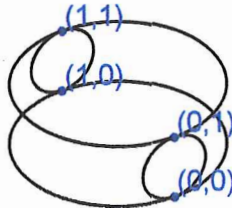
Now factorize the lattice $\mathbb{L}^n$ so that the lattice nodes are factorized using the canonical homomorphism $\mathbb{Z}^n \rightarrow \mathbb{Z}^n/\mathbb{Z}^n$, and the edges with using an equivalence relation based on the parity of edges. As a result of this factorization of the integer lattice, we get the factor lattice $\mathbb{L}^n/\mathbb{Z}^n = \{\mathbb{Z}^n/\mathbb{Z}^n; (\mathbb{Z}, \dots, \mathbb{R}/2\mathbb{Z}, \dots, \mathbb{Z})\}$ with one quotient node and n pairs of quotient edges (semicircles). The group of topological automorphisms of the quotient lattice $\mathbb{L}^n/\mathbb{Z}^n$ that keep the zero point fixed is also equal to $S_2 \wr S_n$. This follows from the fact that there are no topological obstacles for permutations of pairs of quotient edges forming a group of substitutions S_n and for permutations forming a group of substitutions S_2 inside each pair of quotient edges. In turn, if $\mathbb{L}^n/* = \{\mathbb{Z}^n/*; (\mathbb{Z}, \dots, \mathbb{R}/2\mathbb{Z}, \dots, \mathbb{Z})\}$, then the group of topological automorphisms of the quotient lattice $\mathbb{L}^n/*$ is given by the group of topological automorphisms of the bundle of semicircles obtained as a result of compactification (folding) of the lattice axes.

Let us now establish the group of discrete rotations of the quotient lattice $\mathbb{L}^n/A\mathbb{Z}^n$, i.e. the group of topological automorphisms of the quotient lattice that keep its zero point fixed. Recall that the group of discrete rotations of the quotient lattice $\mathbb{L}^1/2\mathbb{Z}$ is trivial and equal to CP_1 . If we form a quotient lattice $\mathbb{L}^2/A\mathbb{Z}^2$ isomorphic to a bundle of two circles intersecting at their two opposite points, then, as we established in the introduction, the group of topological automorphisms of this bundle is equal to the group CP_2 . Finally, let us give a



quotient lattice $\mathbb{L}^n/A\mathbb{Z}^n$ isomorphic to a bundle of n circles intersecting at their two opposite points, and the tangent vectors to the circles at these points form a system of n linearly independent vectors. Since every pair of its 1-dimensional elements is equal to $\mathbb{L}^2/\mathbb{Z}^2$, then every topological automorphism of the quotient lattice is $\mathbb{L}^n/A\mathbb{Z}^n$ it is possible to decompose into a composition of topological automorphisms of all its possible pairs, and therefore a group of topological automorphisms the lattice quotient $\mathbb{L}^n/A\mathbb{Z}^n$ is equal to the parity group CP_n .

Let's give a quotient lattice $\mathbb{L}^2/B\mathbb{Z}^2$, which can be represented as four circles whose intersection points are located at the nodes of the forming parallelogram, and its axial constraint is a bundle of two circles with two semicircle closure points. Hence, as we established in the introduction, topological automorphisms of the



quotient lattice $\mathbb{L}^2/B\mathbb{Z}^2$ make up the group BP_2 . Similarly, let's give a quotient lattice $\mathbb{L}^n/B\mathbb{Z}^n$, which can be represented as an n -parallelepiped composed of 2^n

nodes connected by circles. Then, the topological automorphisms of the quotient lattice $\mathbb{L}^n/B\mathbb{Z}^n$ are generated exclusively by its two-dimensional bundles of circles, and therefore constitute the group BP_n .

Finally, let's give a quotient lattice $\mathbb{L}^n/J\mathbb{Z}^n$, which can be represented as a generative m-parallelepiped composed of 2^m nodes connected by bundles of n_i circles. Since the topological automorphisms of the quotient lattice $\mathbb{L}^n/J\mathbb{Z}^n$ are reduced to the topological automorphisms of an m-parallelepiped with edges in the form of circles (that is, pairs of halves of circles) that make up the group BP_m , which act on the direct product topological automorphisms of the circle bundles that make up the group $\prod^m CP_{n_i}$, then its group of topological automorphisms is equal to $\prod^m CP_{n_i} \rtimes BP_m$. Thus, it is proved

3.1 Proposition. *The group of topological automorphisms of the lattice $\mathbb{L}^n/J\mathbb{Z}^n$ is equivalent to the parity group JP_n*

If we consider a continuum of continuous rotations factor-lattices $\mathbb{L}^n/A\mathbb{Z}^n$ leaving nodal points in place, we get a group of motions of the sphere S^n , leaving its poles fixed, i.e., the group $SO(n)$. Similarly, one could obtain a group of motions of the direct products of spheres that leave fixed nodal points corresponding factor-lattice. Indeed, if the factor is a lattice $\mathbb{L}^n/J\mathbb{Z}^n$ treated as a framework (one-dimensional cell complex) of the direct product of spheres $S^{n_1} \times \dots \times S^{n_m}$, then a group of discrete the rotations of the factor-lattice JP_n could be extended to a group whether parity $SO(n_1, \dots, n_m)$ and map it to a group for a direct product of spheres $S^{n_1} \times \dots \times S^{n_m}$, preserving the node points of the corresponding factor-lattices.

4 Lie algebras of rotating tori

This section does not classify lie algebras generated by tangent linear vector fields of rotating tori, but describes in detail the constructions of some of them. First of all, we present Lie algebras of tangent vector fields of circles. Let's say we have a vector field

$$I = x_2 \partial x_1 - x_1 \partial x_2 \quad (4.1)$$

tangent to Euclidean circles. Then setting this formative, together with the construction of a bracket of vector fields, solves the problem. And since every Lie algebra of linear vector fields is isomorphic to the corresponding matrix algebra

with a commutation operation, we write the generator of this algebra as

$$I = 1_{ij} - 1_{ji} \quad (4.2)$$

where 1_{ij} is a matrix with a single non-zero (equal to one) element at the intersection of i -th row and j -th column, and $i < j$, and in this case $i, j = 1, 2$. Similarly, if we have a vector field

$$J = x_2 \partial x_1 + x_1 \partial x_2, \quad J = 1_{ij} + 1_{ji} \quad (4.3)$$

tangent to pseudo-Euclidean circles, then this is sufficient to specify the corresponding Lie algebra. In turn, the vector field

$$IJ = x_i \partial x_1 - x_2 \partial x_2, \quad D = IJ = 1_{ii} - 1_{jj} \quad (4.4)$$

generates a Lie algebra tangent to pseudo-Euclidean circles in an isotropic basis. Now let's combine all these vector fields $\{I, J, D\}$ to form the basis of the Lie algebra $sl_2(\mathbb{R})$, tangent to a pseudo-Euclidean circle, each point of which rotates along the trajectory of the Euclidean circle. And bearing in mind the factorization of the pseudo-Euclidean plane in the torus $S^1 \times S^1$, we can say that the Lie algebra $sl_2(\mathbb{R})$ is implemented as Euclidean and pseudo-Euclidean rotations of the classical torus wound on the surface of the classical sphere S^2 with poked poles. This construction is easily generalized to form the basis of the Lie algebra $sl_n(\mathbb{R})$. Indeed, if you set a set of matrices

$$\begin{aligned} I_{ij} &= 1_{ij} - 1_{ji} \\ J_{ij} &= 1_{ij} + 1_{ji} \end{aligned} \quad (4.5)$$

where $i < j$ and $i, j = 1, \dots, n$, and matrices

$$D_{ii} = 1_{ii} - 1_{nn} \quad (4.6)$$

where $i = 1, \dots, n-1$, then we get a linearly independent basis $\{I_{ij}, J_{ij}, D_{ii}\}$ of the Lie algebra $sl_n(\mathbb{R})$, which is implemented as rotations of the torus $S^1 \times \dots \times S^n$ over the surface of the sphere S^n with the poles knocked out. Moreover, using paired rotations, you can get Lie algebras $sl_n(\mathbb{C})$. In fact, let

$$\begin{aligned} I_{ij} &= (1_{2i-1, 2j-1} - 1_{2j-1, 2i-1}) + (1_{2i, 2j} - 1_{2j, 2i}) \\ J_{ij} &= (1_{2i-1, 2j-1} + 1_{2j-1, 2i-1}) + (1_{2i, 2j} + 1_{2j, 2i}) \end{aligned} \quad (4.7)$$

where $i < j \in i, j = 1, \dots, n$, and

$$D_{ii} = (1_{2i-1, 2i-1} - 1_{2n-1, 2n-1}) + (1_{2i, 2i} - 1_{2n, 2n}) \quad (4.8)$$

where $i = 1, \dots, n-1$, and

$$I = \sum_1^n (1_{2i-1, 2i} - 1_{2i, 2i-1}) \quad (4.9)$$

Then the set $\{I_{ij}, J_{ij}, D_{ii}, II_{ij}, IJ_{ij}, ID_{ii}\}$ is linearly independent basis for the algebra $sl_n(\mathbb{C})$ and the set $\{I_{ij}, IJ_{ij}, ID_{ii}\}$ forms a basis of the algebra $su(n)$, and $\{sl_n(\mathbb{C})\} = \{su(n)\} + I\{su(n)\}$. The Lie algebra $sl_n(\mathbb{C})$ is implemented as the proper motions of the torus $T^n = S^1 \times \dots \times S^1$ over Villarso circles (due to paired rotations of the torus in a $2n$ - dimensional Euclidean space) and as the motions of this torus over the surface of the hypersphere of a $2n$ -dimensional space with a neutral metric.

The original simplest construction of the Lie algebra is also generalized when we consider the tangent vector fields of a hypersphere of a $2n$ - dimensional space with a neutral metric that rotates in a $2n$ - dimensional space with a Euclidean metric. Moreover, since at the intersection of the hypersphere and the sphere in $2n$ -dimensional space lies a torus (the product of spheres $S^{n-1} \times S^{n-1}$), we will also talk about the Lie algebra of tangent vector fields of rotating tori. Generators of the Lie algebra of vector fields tangent to hyperspheres of a pseudo-Euclidean space with the metric signature $(-n, n)$ are implemented using matrices

$$\begin{aligned} I_{ij}^1 &= 1_{ij} - 1_{ji} \\ I_{ij}^2 &= 1_{i+n, j+n} - 1_{j+n, i+n} \end{aligned} \quad (4.10)$$

where $i < j \in i, j = 1, \dots, n$, and a set of matrices

$$J_{ij} = 1_{i, j+n} + 1_{j+n, i} \quad (4.11)$$

where $i, j = 1, \dots, n$, and Lie algebra generators of vector fields tangent to spheres of $2n$ - dimensional Euclidean space are implemented using matrices

$$1_{ij} - 1_{ji} \quad (4.12)$$

where $i < j$ and $i, j = 1, \dots, 2n$, or (equivalently) using a set of matrices

$$\begin{aligned} I_{ij}^1 &= 1_{ij} - 1_{ji} \\ I_{ij}^2 &= 1_{i+n, j+n} - 1_{j+n, i+n} \end{aligned} \quad (4.13)$$

where $i < j$ and $i, j = 1, \dots, n$, and the matrix

$$I_{ij}^3 = 1_{i, j+n} - 1_{j+n, i} \quad (4.14)$$

where $i, j = 1, \dots, n$. Note that the Euclidean and pseudo-Euclidean space generators together form a new set of matrices

$$D_{ij} = I^3 J = 1_{ij} - 1_{j+n, j+n} \quad (4.15)$$

where $i, j = 1, \dots, n$. Thus, the lie algebra generators of vector fields tangent to the torus $S^{n-1} \times S^{n-1}$ rotating in a $2n$ -dimensional space with Euclidean and neutral metrics are implemented using the set $\{I_{ij}^1, I_{ij}^2, I_{ij}^3, J_{ij}, D_{ij}\}$, consisting of $4n^2 - n$ matrices. Note also that the lie algebra $\left\langle \{I_{ij}^1, I_{ij}^2, I_{ij}^3, J_{ij}, D_{ij}\} \right\rangle_{\mathbb{R}}$ includes the Lie algebra $so_{2n}(\mathbb{R})$ and $sl_n(\mathbb{C})$, and its dimension at $n = 8$ coincides with the dimension of the exclusive Lie algebra e_8 .

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ON CERTAIN ALGEBRAIC STRUCTURES
ASSOCIATED WITH SYMMETRIC SPACES

Noriaki Kamiya
Center for Mathematical Sciences
University of Aizu, 965-8580, Japan
shigekamiya@outlook.jp

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Abstract

The paper will discuss a characterization and a construction of Lie (super)algebras from (ϵ, δ) -Freudenthal Kantor triple systems of certain types and we exhibit examples of such triple systems. From these results, we will show to construct symmetric (super)spaces associated with several simple Lie (super)algebras.

Key words: symmetric spaces, triple systems, Lie (super)algebras.

Introduction

This note is to deal with a study of new ideas associated with symmetric spaces and to describe a survey which conclude a certain history of Jordan algebras in nonassociative algebras with respect to our research works mainly. And also this work is in close contact with a symmetric space with complex structure. That is, in related with curvature and torsion tensors of differential geometry.

From mathematical history's viewpoint, the concept discussed here first appeared with a class of nonassociative algebras, that is commutative Jordan algebras, which was the defining subspace g_{-1} in the Tits-Kantor-Koecher (for short TKK) construction of 3-graded Lie algebras $g = g_{-1} \oplus g_0 \oplus g_1$, such that $[g_i, g_j] \subseteq g_{i+j}$. Nonassociative algebras are rich in algebraic structures, and they provide an important common ground for various branches of mathematics, not only for pure algebra and differential geometry, but also for representation theory and algebraic geometry (for example, [11], [46], [59], [61], [64]). Specially, the concept of nonassociative algebras such as Jordan and Lie (super)algebras plays an important role in many mathematical and physical subjects ([5], [10]-[13], [15], [22], [26], [28], [29], [33], [34], [35], [45], [54], [55], [60], [65]). We also note that the construction and characterization of these algebras can be expressed in terms of the notion of triple systems ([1]-[4], [6]-[8], [20], [23], [24], [40]-[45], [50]-[53], [56]-[58]) by using the standard embedding method ([22], [48], [49], [57], [63]). In particular, the generalized Jordan triple system of second order, or $(-1, 1)$ -Freudenthal Kantor triple system (for short $(-1, 1)$ -FKTS), is a useful concept ([13]-[21], [41]-[44], [47], [62]) for the constructions of simple Lie algebras, while the $(-1, -1)$ -FKTS plays the same role ([6], [22], [25], [27], [32], [33]) for the construction of Lie superalgebras, while the δ -Jordan Lie triple systems act similarly for that of Jordan superalgebras ([23], [24], [56]). Specially, we have constructed a model of Lie superalgebras $D(2, 1; \alpha)$, $G(3)$ and $F(4)$ ([22], [25], [27]).

As a final comment of this introduction, we provide well-known results due to O. Loos as follows; if A is a unital commutative Jordan algebra with a unital element e , that is, satisfying $(xy)x^2 = x(y(x^2))$, and $xy = yx$, then the triple product given by

$$\{xyz\} = (xy)z + x(yz) - y(xz)$$

defines a Jordan triple system with $\{xey\} = xy$, i.e., it satisfies the two relations $\{xy\{abc\}\} = \{\{xya\}bc\} - \{a\{yxb\}c\} + \{ab\{xyc\}\}$ (this relation is often called a fundamental identity), and $\{xyz\} = \{zyx\}$ (this relation is called a commutative identity, since $xy = \{xey\} = \{yex\} = yx$) and next the new triple product $[xyz]$ given by

$$[xyz] = \{xyz\} - \{yxz\}$$

defines a Lie triple system.

Briefly summarizing this article, we will generalize these results and exhibit examples of Lie (super)algebras associated with generalized Jordan triple systems. Toward to its applications, in particular, we will give a construction of symmetric spaces with an almost complex structure.

Roughly describing, we have an illustration for our concept ;

Algebraic structures $\Longleftrightarrow$ Geometric structures.

For examples, there are certain algebraic structures associated with symmetric, R-symmetric, homogeneous spaces, totally geodesic manifold, and symmetric domains, etc.

1 Preamble and Definitions

In this paper triple systems have finite dimension being defined over a field Φ of characteristic $\neq 2$ or 3 , unless otherwise specified. In order to render the paper as self-contained as possible, we recall first the definition of a generalized Jordan triple system of second order (for short GJTS of 2nd order).

A vector space V over a field Φ endowed with a trilinear operation $V \times V \times V \rightarrow V$, $(x, y, z) \mapsto (xyz)$ is said to be a *GJTS of 2nd order* if the following conditions are fulfilled:

$$(ab(xyz)) = ((abx)yz) - (x(bay)z) + (xy(abz)), \quad (1)$$

$$K(K(a, b)x, y) - L(y, x)K(a, b) - K(a, b)L(x, y) = 0, \quad (2)$$

where $L(a, b)c := (abc)$ and $K(a, b)c := (acb) - (bca)$.

A *Jordan triple system* (for short JTS) satisfies (1) and the following condition

$$(abc) = (cba), \text{ i.e., } K(a, c)b = 0. \quad (3)$$

The JTS is a special case in the GJTS of 2nd order since $K(x, y) \equiv 0$.

We next can generalize the concept of GJTS of 2nd order as follows (see [13], [14], [18], [22], [28], [36] [63] and the earlier references therein).

For $\varepsilon = \pm 1$ and $\delta = \pm 1$, a triple product that satisfies the identities

$$(ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)), \quad (4)$$

$$K(K(a, b)x, y) - L(y, x)K(a, b) + \varepsilon K(a, b)L(x, y) = 0, \quad (5)$$

where

$$L(a, b)c := (abc), \quad K(a, b)c := (acb) - \delta(bca), \quad (6)$$

is called an (ε, δ) -*Freudenthal - Kantor triple system* (for short (ε, δ) -FKTS). An (ε, δ) -FKTS is said to be *unitary* if $Id \in \{K(a, b)\}_{span}$.

A triple system satisfying only the identity (4) is called a *generalized FKTS* (for short GFKTS), while the identity (5) is called the *second order condition*.

Remark From the relation Eq. (6), we note that

$$K(b, a) = -\delta K(a, b). \quad (7)$$

A triple system is called a (α, β, γ) *triple system associated with a bilinear form* if

$$(xyz) = \alpha \langle x, y \rangle z + \beta \langle y, z \rangle x + \gamma \langle z, x \rangle y,$$

where $\langle x, y \rangle$ is a bilinear form such that $\langle x, y \rangle = \kappa \langle y, x \rangle$, $\kappa = \pm 1$, $\alpha, \beta, \gamma \in \Phi$.

From now on we will mainly consider this type of triple system.

An (ε, δ) -FKTS is said to be *balanced* if there is a bilinear form $\langle x, y \rangle \in \Phi^*$ such that $K(x, y) = \langle x, y \rangle Id$, that is, $\dim \{K(x, y)\}_{span} = 1$ holds.

Remark We note that a balanced triple system (i.e., it fulfills $K(x, y) = \langle x, y \rangle Id$) is unitary, since $Id \in \{K(x, y)\}_{span}$.

Triple products are denoted by (xyz) , $\{xyz\}$, $[xyz]$ and $\langle xyz \rangle$ upon their suitability.

Remark We note that the concept of GJTS of 2nd order coincides with that of $(-1, 1)$ -FKTS. Thus we can construct the corresponding Lie algebras by means of the standard embedding method ([6], [13]-[19], [21], [22], [25], [27], [43]).

For $\delta = \pm 1$, a triple system $(a, b, c) \mapsto [abc]$, $a, b, c \in V$ is called a δ -Lie triple system (for short δ -LTS) if the following three identities are fulfilled

$$\begin{aligned} [abc] &= -\delta[bac], \\ [abc] + [bca] + [cab] &= 0, \\ [ab[xyz]] &= [[abx]yz] + [x[aby]z] + [xy[abz]], \end{aligned} \quad (8)$$

where $a, b, x, y, z \in V$. An 1-LTS is a LTS while a -1 -LTS is an *anti-LTS*, by ([14]). Note that the set $L(V, V)$ of all left multiplications $L(x, y)$ of V is a Lie subalgebra of $\text{Der } V$, where we denote by $L(x, y)z = [xyz]$.

Proposition 1.1 ([13]-[16], [22]) *Let $(U(\varepsilon, \delta), \langle xyz \rangle)$ be an (ε, δ) -FKTS. If J is an endomorphism of $U(\varepsilon, \delta)$ such that $J \langle xyz \rangle = \langle JxJyJz \rangle$ and $J^2 = -\varepsilon\delta Id$, then $(U(\varepsilon, \delta), [xyz])$ is a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the product*

$$[xyz] := \langle xJyz \rangle - \delta \langle yJxz \rangle + \delta \langle xJzy \rangle - \langle yJzx \rangle. \quad (9)$$

Corollary ([13]) *Let $U(\varepsilon, \delta)$ be an (ε, δ) -FKTS. Then the vector space $T(\varepsilon, \delta) = U(\varepsilon, \delta) \oplus U(\varepsilon, \delta)$ becomes a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the triple product*

$$\left[\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} \right] = \begin{pmatrix} L(a, d) - \delta L(c, b) & \delta K(a, c) \\ -\varepsilon K(b, d) & \varepsilon(L(d, a) - \delta L(b, c)) \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}. \quad (10)$$

Thus we can obtain the standard embedding Lie algebra (if $\delta = 1$) or Lie superalgebra (if $\delta = -1$), $L(U(\varepsilon, \delta)) = D(T(\varepsilon, \delta), T(\varepsilon, \delta)) \oplus T(\varepsilon, \delta)$, associated with $T(\varepsilon, \delta)$ where $D(T(\varepsilon, \delta), T(\varepsilon, \delta))$ is the set of inner derivations of the δ -Lie triple system $T(\varepsilon, \delta)$;

$$D(T(\varepsilon, \delta), T(\varepsilon, \delta)) := \left\{ \begin{pmatrix} L(a, b) & \delta K(c, d) \\ -\varepsilon K(e, f) & \varepsilon L(b, a) \end{pmatrix} \right\}_{span},$$

$$T(\varepsilon, \delta) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x, y \in U(\varepsilon, \delta) \right\}_{\text{span}}.$$

We use the following notation:

$$\mathbf{k} := \{K(x, y) \in \text{End } U(\varepsilon, \delta) | x, y \in U(\varepsilon, \delta)\} \text{ and}$$

$$\{EFG\} := EFG + GFE, \forall E, F, G \in \mathbf{k}.$$

Then, we may make the structure of a JTS $\mathbf{k}$ with respect to the triple product $\{EFG\} \in \mathbf{k}$ ([20]). Also the JTS $\mathbf{k}$ is called *nondegenerate* if $K(x, y) = 0$ for any $y \in U(\varepsilon, \delta)$ implies $x = 0$. Hence we have the following Proposition.

Proposition 1.2 ([15], [31]) *Let U be a unitary (ε, δ) -FKTS and $L(U)$ be the standard embedding Lie (super)algebra associated with U . Then the following are equivalent:*

- (i) U is simple,
- (ii) the Lie (super)algebra $L(U)$ is simple,
- (iii) the JTS $\mathbf{k} := \{K(a, b)\}_{\text{span}}$ is simple and nondegenerate.

Remark We note that $L(U) = L(U(\varepsilon, \delta)) := L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ is the five graded Lie (super)algebra such that $U(\varepsilon, \delta) \oplus U(\varepsilon, \delta) = L_{-1} \oplus L_1 = T(\varepsilon, \delta)$ (δ -LTS), $L_{-2} = \mathbf{k}$ (JTS) and $D(T(\varepsilon, \delta), T(\varepsilon, \delta)) = L_{-2} \oplus L_0 \oplus L_2$ (the derivation of $T(\varepsilon, \delta)$) equipped with $[L_i, L_j] \subseteq L_{i+j}$ and $L_{-1} \oplus L_1 = L(U)/L_{-2} \oplus L_0 \oplus L_2$. In Introduction, we had used the notation $g = g_{-1} \oplus g_0 \oplus g_1$ instead of $L_{-1} \oplus L_0 \oplus L_1$. This Lie (super)algebra construction is one of reasons to study nonassociative algebras and triple systems without using root systems (for a Lie superalgebra, refer to ([9], [12], [60])).

2 A generalized curvature and torsion tensors

Let $L = L(U(\varepsilon, \delta)) = L(W, W) \oplus W$ be the Lie (super)algebra defined from a δ LTS as in section one, that is, the δ -LTS $W = T(\varepsilon, \delta) = L_{-1} \oplus L_1$ is induced from $L_{-1} = U(\varepsilon, \delta)$ (as L_{-1} has the structure of a (ε, δ) -FKTS).

We introduce a generalization of covariant derivative ∇ in differential geometry as follows; $\nabla : L \rightarrow \text{End } L$ defined by

$$\nabla_X Y = [X, Y] = -\delta[Y, X],$$

$$\begin{aligned}\nabla_X[Y, Z] &= [YZX] = -\delta[ZYX], \\ \nabla_{[X, Y]}Z &= -[XYZ] = -\delta[YXZ], \\ \nabla_{[X, Y]}[V, Z] &= [[V, Z][X, Y]] = -\delta[[X, Y][V, Z]], \\ \text{for any } X, Y, Z &\in W.\end{aligned}$$

Furthermore, a generalized curvature tensor defined by

$$C_\delta(X, Y) = \nabla_X \nabla_Y - \delta \nabla_Y \nabla_X - \nabla_{[X, Y]} \quad (11)$$

is identically zero, i.e., $C_\delta(X, Y) = 0$ in L , for any $X, Y \in W$. Indeed, we demonstrate the proof below.

First we calculate

$$\begin{aligned}C_\delta(X, Y)Z &= (\nabla_X \nabla_Y - \delta \nabla_Y \nabla_X)Z - \nabla_{[X, Y]}Z \\ &= \nabla_X[Y, Z] - \delta \nabla_Y[X, Z] + [XYZ] \\ &= [YZX] - \delta[XZY] + [XYZ] \\ &= [YZX] + [ZXY] + [XYZ] = 0.\end{aligned}$$

Second, it follow

$$\begin{aligned}C_\delta(X, Y)[V, Z] &= (\nabla_X \nabla_Y - \delta \nabla_Y \nabla_X)[V, Z] - \nabla_{[X, Y]}[V, Z] \\ &= [X, [VZY]] - \delta[Y, [VZX]] + \delta[[X, Y], [V, Z]] \\ &= [X, L(V, Z)Y] - \delta[Y, L(V, Z)X] - L(V, Z)[X, Y] = 0\end{aligned}$$

(by $[Y, L(V, Z)X] = -\delta[L(V, Z)X, Y]$ and $[[X, Y], [V, Z]] = -\delta[[V, Z], [X, Y]]$)
for any $X, Y, Z, V \in T(\varepsilon, \delta)$.

However a generalized torsion tensor defined by

$$S_\delta(X, Y) = \nabla_X Y - \delta \nabla_Y X - [X, Y] \quad (12)$$

is not zero, since it gives $S_\delta(X, Y) = [X, Y] - \delta[Y, X] - [X, Y] = [X, Y]$.

To later section we next define the Nijenhuis operator

$$N(X, Y) = [JX, JY] + J^2[X, Y] - J[JX, Y] - J[X, JY],$$

where J is an almost complex structure on W , this concept associated with the case of $\delta = 1$ is appeared in ([36]).

3 Examples of (ε, δ) -JTS

We consider here examples of the particular case defined by bilinear forms $\langle x, y \rangle$, that is, of an (ε, δ) -JTS, or of (α, β, γ) triple systems equipped with $K(x, y) \equiv 0$.

Example 3.1 Let V be a vector space with a symmetric bilinear form $\langle x, y \rangle$. Then

$$\langle xyz \rangle = \langle x, y \rangle z + \langle y, z \rangle x - \langle z, x \rangle y$$

defines on V a $(-1, 1)$ -JTS.

Note that $(-1, 1)$ -JTS is same as the JTS.

Example 3.2 Let V be a vector space with an anti-symmetric bilinear form $\langle x, y \rangle$. Then

$$\langle xyz \rangle = \langle x, y \rangle z + \langle y, z \rangle x - \langle z, x \rangle y$$

defines on V a $(1, -1)$ -JTS.

Example 3.3 Let V be a vector space with a symmetric bilinear form $\langle x, y \rangle$. Then

$$\langle xyz \rangle = \langle x, y \rangle z - \langle y, z \rangle x$$

defines on V a $(-1, -1)$ -JTS.

Example 3.4 Let V be a vector space with an anti-symmetric bilinear form $\langle x, y \rangle$. Then

$$\langle xyz \rangle = \langle x, y \rangle z - \langle y, z \rangle x$$

defines on V a $(1, 1)$ -JTS.

Example 3.5 Let V be a set of alternative matrix $Asym(n, \Phi) = \{x | {}^t x = -x\}$. Then

$$\langle xyz \rangle = x {}^t y z - \varepsilon z {}^t y x, \text{ where } \forall x, y, z \in V$$

defines on V a $(\varepsilon, -\varepsilon)$ JTS.

Proposition 3.1 Let $(U, \langle xyz \rangle)$ be an (ε, δ) -JTS. Then the triple system is a δ -LTS with respect to the new product

$$[xyz] = \langle xyz \rangle - \delta \langle yxz \rangle. \quad (13)$$

In the next section 5 subsection we study the case of an (ε, δ) -FKTS, but we give first two examples which are not (ε, δ) -JTS as it follows.

Proposition 3.2 *Let $(U, \langle xyz \rangle)$ be a triple system with $\langle xyz \rangle = \langle y, z \rangle x$ and $\langle x, y \rangle = -\varepsilon \langle y, x \rangle$. Then this triple system is an (ε, δ) -FKTS.*

Proposition 3.3 ([16], [18]) *Let U be a balanced $(1, 1)$ -FKTS satisfying $\langle \langle xxx \rangle, x \rangle \equiv 0$ (identically) and $\langle x, y \rangle$ is nondegenerate. Then U has a triple product defined by*

$$\langle xyz \rangle = \frac{1}{2}(\langle y, x \rangle z + \langle y, z \rangle x + \langle x, z \rangle y). \quad (14)$$

On the other hand, note that the balanced $(1, 1)$ -FKTS induced from an exceptional Jordan algebra is closely related to the 56 dimensional meta symplectic geometry due to H. Freudenthal ([13], [15], [16], [18] and the earlier references therein). Also the correspondence of a quaternionic symmetric space and the balanced $(1, 1)$ FKTS has been studied in ([5]).

4 Symmetric spaces associated with (ε, δ) Jordan triple systems

This article (section) means that the symmetric space in sense of Prof. Wolfgang Bertram [Lecture Note in Math. "The geometry of Jordan and Lie structures" Springer, vol.1754 (2000)]. In this book, the following is described;

a) the category of germs of symmetric space with invariant almost complex structure is equivalent to the category of Lie triple systems with invariant complex structure,

b) the category of germs of symmetric space with invariant polarizations is equivalent to the category of Lie triple system with invariant polarization.

Hence from these results, it seems that it is important to construct a Lie triple system from a Jordan triple system (abbreviated JTS). Therefore we construct δ -Lie triple systems associated with a $(\varepsilon, -\varepsilon)$ -JTS of more a general case (when $\varepsilon\delta = -1$).

From now on, let V be a (ε, δ) -JTS with $\varepsilon = -\delta$, that is, $\varepsilon\delta = -1$ and $\widehat{W}$ be a subset in the δ -LTS $W = T(\varepsilon, \delta)$ satisfying

$$\widehat{W} := \widehat{W}_+ \cup \widehat{W}_-, \text{ where } \widehat{W}_+ := \left\{ \begin{pmatrix} x \\ x \end{pmatrix} \middle| x \in V \right\}, \widehat{W}_- := \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \middle| x \in V \right\}.$$

If we set $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $i = \sqrt{-1}$, then we get $JW_+ = W_-$, $JW_- = W_+$.

From a special case of section one, for δ -LTS W with $K(x, y) \equiv 0$, we have

$$\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right] = \begin{pmatrix} (L(x_1, y_2) - \delta L(y_1, x_2))z_1 \\ (\varepsilon L(y_2, x_1) - \varepsilon \delta L(x_2, y_1))z_2 \end{pmatrix}.$$

Thus for $\widehat{W}_+$ and $\widehat{W}_-$ respectively, we obtain

$$\left[\begin{pmatrix} x \\ x \end{pmatrix} \begin{pmatrix} y \\ y \end{pmatrix} \begin{pmatrix} z \\ z \end{pmatrix} \right] = \begin{pmatrix} L(x, y)z - \delta L(y, x)z \\ \varepsilon L(y, x)z - \varepsilon \delta L(x, y)z \end{pmatrix} \in \widehat{W}_+,$$

$$\left[\begin{pmatrix} x \\ -x \end{pmatrix} \begin{pmatrix} y \\ -y \end{pmatrix} \begin{pmatrix} z \\ -z \end{pmatrix} \right] = \begin{pmatrix} -L(x, y)z + \delta L(y, x)z \\ \varepsilon L(y, x)z - \varepsilon \delta L(x, y)z \end{pmatrix} \in \widehat{W}_-.$$

Also, we have $[\widehat{W}_- \widehat{W}_- \widehat{W}_+] \subset \widehat{W}_+$ and $[\widehat{W}_+ \widehat{W}_+ \widehat{W}_-] \subset \widehat{W}_-$ etc.

Here we define for $X, Y, Z \in \widehat{W}_+$ or $X, Y, Z \in \widehat{W}_-$,

$$R(X, Y)Z = -[X, Y, Z] = -[[X, Y], Z],$$

$$T(X, Y)Z := T(X, Y, Z) = -\frac{1}{2}(R(X, Y)Z - JR(X, J^{-1}Y)Z).$$

Hence, for example $T(X, Y)Z = \begin{pmatrix} -L(x, y)z \\ L(x, y)z \end{pmatrix}$ for any $X, Y, Z \in \widehat{W}_-$.

Then we have the following.

Theorem 4.1. *Under the above assumption and $\sqrt{-1} \in \Phi$ (the base field), we have*

- i) $\widehat{W}_+$ and $\widehat{W}_-$ are a δ -LTS with respect the product $[XYZ]$,
- ii) $\widehat{W}_+$ and $\widehat{W}_-$ are a $(\varepsilon, -\varepsilon)$ -JTS with respect to the product $T(X, Y, Z)$,
- iii) $\widehat{W}$ is twisted (i.e., $[JXYZ] = -[XJYZ]$) and almost complex with J ,

iv) $N(X, Y)$ is vanished, where $N(X, Y) := [JX, JY] + J^2[X, Y] - J[JX, Y] - J[X, JY]$, (Nijenhuis operator).

Note that $JT(X, Y, Z) = T(JX, Y, Z) = -T(X, JY, Z)$ hold for $X, Y, Z \in \widehat{W}_\pm$, i.e., this concept is hermite in the sense of W. Bertram.

Remark Note that there is no the addition on $\widehat{W} := \widehat{W}_+ \cup \widehat{W}_-$, but there is the multiplication $[XYZ]$ on $\widehat{W}$ and the addition on $\widehat{W}_+$ or $\widehat{W}_-$.

Similar if $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. Then we have $\widehat{W}$ is twisted and polarized, i.e., $J^2 = Id$ on $\widehat{W}$.

If $J = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, then it is straight with $T(X, Y) = 0$, i.e., $[JXYZ] = [XJYZ]$, if $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then it is straight with $T(X, Y) = 0$, similarly.

Thus we note that there are the concept of invariant (i.e., $J[XYZ] = [XYJZ]$), automorphism and derivation on $\widehat{W}_\pm$.

Furthermore, if $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \Rightarrow N(X, Y) = 0$ (identically zero),

if $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \Rightarrow N(X, Y) \neq 0$,

if $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \Rightarrow$ auto, but $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is not auto,

if $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \Rightarrow$ auto, but $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is not auto.

Sumarizing these results, for the structure J of $\widehat{W}$ we have the table as follows, ($\varepsilon = -1, \delta = 1$)

	almost complex	polarized
twisted auto	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ (type I)	$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$
twisted anti-auto	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (type II)	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
straight	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ or $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ $T(X, Y) \equiv 0$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $T(X, Y) \equiv 0$

where the anti-automorphism is denoted by $[JXJYJZ] = -J[XYZ]$ on $\widehat{W}$.

Note that J is twisted and invariant $\implies$ derivation.

5 Examples of Lie (super)algebras associated with (ε, δ) Freudenthal-Kantor triple systems and B_3 type Lie algebra

Unless otherwise stated, all Lie (super)algebras considered here are complex and finite dimensional. Note that the examples of symmetric spaces induced from Jordan triple systems are to refer W. Brtram's book, hence this section deals without the cases.

5.1 Examples of simple Lie superalgebra

Example a) $C(n+1)$ type is of dimension $\dim C(n+1) = 2n^2 + 5n + 1$.

Let U be the set of matrices $M(1, 2n; \Phi)$. Then, by Example 3.2, it follows that the triple product

$$L(x, y)z = \langle xyz \rangle := \langle x, y \rangle z + \langle y, z \rangle x - \langle z, x \rangle y$$

such that the bilinear form fulfills $\langle x, y \rangle = -\langle y, x \rangle$, is a $(1, -1)$ -JTS, since $K(x, y) \equiv 0$ (identically). Furthermore, the standard embedding Lie superalgebra is 3-graded and of $C(n+1)$ type. For the extended Dynkin diagram, we obtain

$$\begin{aligned} L_{-1} \oplus L_0 \oplus L_1 &:= \left\{ \begin{pmatrix} L(a, b) & 0 \\ 0 & \varepsilon L(b, a) \end{pmatrix} \middle| \varepsilon = 1 = -\delta \right\}_{span} \oplus \left\{ \begin{pmatrix} e \\ f \end{pmatrix} \right\}_{span} \cong \\ &\quad \otimes \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_n \quad \alpha_{n+1} \\ &\quad \parallel > \quad \circ - \circ - - - - \circ < = \circ \\ &\quad \otimes \alpha_0 \\ &= C(n+1) \text{ type } (\alpha_1 \otimes \text{deleted}). \end{aligned}$$

Also, we obtain

$$L_0 := \left\{ \left(\begin{array}{cc} L(a, b) & 0 \\ 0 & \varepsilon L(b, a) \end{array} \right) \middle| \varepsilon = 1 = -\delta \right\}_{span} \cong$$

$$\begin{array}{ccccccc} \alpha_2 & \alpha_3 & & & \alpha_n & \alpha_{n+1} & \\ \circ & - & \circ & - & - & - & \circ \leq \circ \end{array}$$

$$= C_n \oplus \Phi Id (\alpha_1 \otimes \text{ and } \alpha_0 \otimes \text{ deleted}).$$

Thus the last diagram is obtained from the extended Dynkin diagram of $C(n+1)$ type by deleting $\alpha_1 \otimes$ and $\alpha_0 \otimes$.

Example b) $B(n, 1)$ and $D(n, 1)$ type are of dimension $\dim B(n, 1) = 2n^2 + 5n + 5$ and $\dim D(n, 1) = 2n^2 + 3n + 3$, respectively.

Let U be the set of matrices $M(1, l; \Phi)$. Then, by straihtfoward calculations, it follows that the triple product

$$L(x, y)z = \langle xyz \rangle := \frac{1}{2}(\langle x, y \rangle z - \langle y, z \rangle x + \langle z, x \rangle y)$$

such that the bilinear form fulfills $\langle x, y \rangle = \langle y, x \rangle$ is a $(-1, -1)$ -FKTS. Furthermore, the standard embedding Lie superalgebra is 5-graded and of $B(n, 1)$ type if $l = 2n + 1$, or of $D(n, 1)$ type if $l = 2n$. For the extended Dynkin diagram, we obtain from the results of § 1 the following.

For the case of $B(n, 1)$ type we have

$$L_{-2} \oplus L_0 \oplus L_2 := D(T(-1, -1), T(-1, -1)) =$$

$$\left\{ \left(\begin{array}{cc} L(a, b) & \delta K(c, d) \\ -\varepsilon K(e, f) & \varepsilon L(b, a) \end{array} \right) \middle| \varepsilon = -1 = \delta \right\}_{span} \cong$$

$$\begin{array}{ccccccc} \alpha_0 & \alpha_1 & \alpha_2 & & \alpha_n & \alpha_{n+1} & \\ \circ = > \otimes & - & \circ & - & - & - & \circ = > \circ \end{array}$$

$$= A_1 \oplus B_n \text{ type } (\alpha_1 \otimes \text{ deleted}).$$

Also, we obtain

$$L_0 := \left\{ \left(\begin{array}{cc} L(a, b) & 0 \\ 0 & \varepsilon L(b, a) \end{array} \right) \middle| \varepsilon = -1 = \delta \right\}_{span} \cong$$

Example c) We study the case of $g_{-1} = U = Mat(1, 5; \Phi)$. Hereafter in this subsection, as a reason of traditional notation, we often would like to denote by g_i instead of L_i , ($i = 0, \pm 1, \pm 2$) and by $\{xyz\}$ instead of $\langle xyz \rangle$.

In this case, g_{-1} is a JTS with respect to the product

$$\{xyz\} = x^t yz + y^t zx - z^t xy, \forall x, y, z \in g_{-1}$$

where $^t x$ denotes the transpose matrix of x .

By straightforward calculations, the standard embedding Lie algebra $L(U) = g$ can be shown to be a 3-graded B_3 -type Lie algebra with $g = g_{-1} \oplus g_0 \oplus g_1$ and a LTS $T(U) = g_{-1} \oplus g_1$. Thus, we have

$$g_0 = \text{Der } U \oplus \text{Anti} - \text{Der } U \cong B_2 \oplus \Phi H, \text{ where } H = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}.$$

Here in view of the relations $[S(x, y), L(a, b)] = L(S(x, y)a, b) + L(a, S(x, y)b)$, and $[A(x, y), L(a, b)] = L(A(x, y)a, b) - L(a, A(x, y)b)$ for all $L(a, b) \in \text{End } U$, when $\varepsilon = -1, \delta = 1$, we use the following notations;

$$\text{Der } U := \{L(x, y) - L(y, x)\}_{\text{span}},$$

$$\text{Anti} - \text{Der } U := \{L(x, y) + L(y, x)\}_{\text{span}},$$

$$g_0 = \left\{ \begin{pmatrix} L(x, y) & 0 \\ 0 & -L(y, x) \end{pmatrix} \right\}_{\text{span}} = \{S(x, y) + A(x, y)\}_{\text{span}}$$

where $S(x, y) := L(x, y) - L(y, x) \in \text{Der } U$, $A(x, y) := L(x, y) + L(y, x) \in \text{Anti} - \text{Der } U$.

Example d) Second, we study the case of $g_{-1} = U = \text{Mat}(2, 3; \Phi)$. In this case, g_{-1} is a GJTS of 2nd order (i.e., $(-1, 1)$ -FKTS) with $\dim \{K(x, y)\}_{\text{span}} = 1$ with respect to the product

$$\{xyz\} = x^t yz + z^t yx - z^t xy, \forall x, y, z \in g_{-1}.$$

By straightforward calculations, it can be shown that the standard embedding Lie algebra $L(U) = g$ is a 5-graded B_3 -type Lie algebra with $g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$ and $\dim g_{-2} = \dim g_2 = \dim \{K(x, y)\}_{\text{span}} = 1$. Thus, we have

$$g_0 = \text{Der } U \oplus \text{Anti} - \text{Der } U \cong A_1 \oplus A_1 \oplus \Phi H, \text{ where } H = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}.$$

Furthermore, we obtain a LTS $T(U)$ of $\dim T(U) = \dim (g_{-1} \oplus g_1) = 12$,

$$\text{Der}(g_{-1} \oplus g_1) = g_{-2} \oplus g_0 \oplus g_2 = A_1 \oplus A_1 \oplus A_1 \cong \text{Der } T(U).$$

Also, in this case, we note that $T(U) = L(U)/\text{Der } T(U) = g/(g_{-2} \oplus g_0 \oplus g_2) (= g_{-1} \oplus g_1)$ is the tangent space of a quaternion symmetric space of dimension 12, since $T(U)$ is a Lie triple system associated with g_{-1} .

Example e) Third, we study the case of $g_{-1} = U = \text{Mat}(1, 3; \Phi)$. In this case, g_{-1} is a GJTS of 2nd order (i.e., $(-1, 1)$ -FKTS) with respect to the product

$$\{xyz\} = x^t yz + z^t yx - y^t xz, K(x, y)z = \{xzy\} - \{yzx\}, \forall x, y, z \in g_{-1}.$$

By straightforward calculations, the standard embedding Lie algebra $L(U) = g$ can be shown to be a 5-graded B_3 -type Lie algebra with $g = g_{-2} \oplus \dots \oplus g_2$ and $\dim g_{-2} = \dim g_2 = 3$. Thus, we have

$$g_0 = \text{Der } U \oplus \text{Anti} - \text{Der } U \cong A_2 \oplus \Phi H, g_{-2} = \{K(x, y)\}_{\text{span}} = \text{Alt}(3, 3; \Phi).$$

Furthermore, we obtain a LTS $T(U)$ of $\dim T(U) = \dim (g_{-1} \oplus g_1) = 6$,

$$\text{Der}(g_{-1} \oplus g_1) = g_{-2} \oplus g_0 \oplus g_2 = A_3 \cong \text{Der } T(U).$$

This case $g_{-2} = \{K(x, y)\}_{\text{span}} = \mathbf{k}$ has the structure of a JTS (cf. section 2).

Remark We remark that the cases (a) and (b) (resp. (c), (d), (e)) are $\delta = -1$ (resp. $\delta = 1$).

Remark For the root system $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}$ and the highest root $-\rho = \{\alpha_1 + 2\alpha_2 + 2\alpha_3\}$ of the simple Lie algebra B_3 , the case of (c) means that $g_{-1} = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}$ and $g_{-2} = 0$, the case of (d) means that $g_{-1} = \{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3\}$ and $g_{-2} = \{-\rho\}$, the case of (e) means that $g_{-1} = \{\alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3\}$ and $g_{-2} = \{\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}$.

Example f) Finally, we study the case of $g_{-1} = U = \text{Mat}(1, 7; \Phi)$. In this case, g_{-1} is a JTS (i.e., $(-1, 1)$ -FKTS with $K(x, y) \equiv 0$) with respect to the product

$$\{xyz\} = x^t yz + y^t zx - z^t xy, \forall x, y, z \in g_{-1}.$$

By straightforward calculations, the standard embedding Lie algebra $L(U) = g$ can be shown to be a 3-graded B_4 -type Lie algebra with $g = g_{-1} \oplus g_0 \oplus g_1$ and $\dim g_{-2} = \dim g_2 = 0$. Thus, we have

$$g_0 = \text{Der } U \oplus \text{Anti} - \text{Der } U \cong B_3 \oplus \Phi H,$$

$$\text{Der } U = \{L(x, y) - L(y, x)\}_{\text{span}} = \text{Alt}(7, 7; \Phi) \cong B_3$$

$$\text{Anti} - \text{Der } U = \{L(x, y) + L(y, x)\}_{\text{span}} \cong \Phi H.$$

This case is obtained from $\text{Der } U$ such that $U = \text{Mat}(1, 7; \Phi)$ with the JTS structure and so this derivation $\text{Der } U$ is a subalgebra of the B_4 -type Lie algebra associated with $g_{-1} = U = \text{Mat}(1, 7; \Phi)$.

6 A generalization case of (ε, δ) JTS

Let V be a (ε, δ) -FKTS and $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $i = \sqrt{-1}$. This concept is a generalization of (ε, δ) -JTS in section 4. Then we set $\widehat{W} := \widehat{W}_+ \cup \widehat{W}_-$ satisfying

$$\widehat{W}_+ = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} \middle| x \in V \right\}, \widehat{W}_- = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \middle| x \in V \right\}. \text{ Hence } \widehat{W} \subset W.$$

Theorem 6.1 For the above $\widehat{W}$ and $\varepsilon\delta = -1$, $\sqrt{-1} \in \Phi$, we have

- i) $\widehat{W}_\pm$ is a δ -LTS with respect the product $[XYZ]$ for any $X, Y, Z \in \widehat{W}_\pm$,
- ii) $T(X, Y, Z) = -\frac{1}{2}(R(X, Y)Z - JR(X, J^{-1}Y)Z)$ is a (ε, δ) -FKTS, where $R(X, Y)Z = -[XYZ]$, i.e., $(V, L(x, y)z) \leftrightarrow (\widehat{W}_\pm, T(X, Y, Z))$,
- iii) $N(X, Y)$ is vanished.

Indeed from the relation (10) in section one, for example, for $\widehat{W}_-$,

$$\left[\begin{pmatrix} x \\ -x \end{pmatrix} \begin{pmatrix} y \\ -y \end{pmatrix} \begin{pmatrix} z \\ -z \end{pmatrix} \right] = \begin{pmatrix} -L(x, y)z + \delta L(y, x)z - \delta K(x, y)z \\ \varepsilon L(y, x)z - \varepsilon \delta L(x, y)z - \varepsilon K(x, y)z \end{pmatrix},$$

$[XYZ] \in \widehat{W}_-$ and $JT(X, Y, Z) \in \widehat{W}_+$ for any $X, Y, Z \in \widehat{W}_-$, we obtain the results.

Remark. This generalized concept means that there is a symmetric (super)space associated with the (ε, δ) -FKTS, as same methods in section 4. However the details (type I and II) will be discussed in forthcoming paper.

7 Concluding Remarks

In this section, we give several references of mathematical physics related in our works. That is, we note that there are applications toward the Yang-Baxter equations associated with triple systems ([26], [39], [57]) and also toward the field theory associated with Hermitian triple systems ([37], [38]), furthermore, for other mathematical physics, it seems that the books ([28], [33]) are useful.

In final comments of this section, for another way of mathematical physics, there are a lot of papers with respect to the Lie admissible algebras (in more general, mutation algebras defined by $x * y = \alpha xy + \beta yx$, $\alpha, \beta \in \Phi$) and the quantum mechanics due to Prof. R. M. Santilli in Hadronic Jour. series.

8 Appendix (history from a certain personal viewpoint)

For a mathematical history, we describe belows:

This brief history (with respect to nonassociative algebras) is a story from author's personal aspect (judgement). Triple systems (ternary algebras) have first been appeared from Prof. N. Jacobson and continued by Profs. O. Loos, K. Meyberg and E. Neher of students of Prof. M. Koecher in Germany, also certain triple systems associated with the geometry of 56 dimensional due to Prof. H. Freudenthal have been studied by Prof. J. Faulkner (resp. K. Meyberg) of the student of Prof. N. Jacobson (resp. Prof. M. Koecher).

On the other hand, there is a history;

H. Freudenthal (Netherlands) — > K. Yamaguti (Japan) or I. L. Kantor (Russian and Sweden, he was born in Belarus) — > Author (N. Kamiya) — > D. Mondoc (but these arrows are no students), however, Dr. Mondoc is only a student of Prof. Kantor in Sweden.

Profs. O. Loos and E. Neher in the student of Prof. M. Koecher in Germany are working in Jordan triple systems and Jordan pairs. Profs. Kantor, S. Okubo and author (N. Kamiya) are studying in their generalizations, for example, refer to N. Kamiya and S. Okubo "Representation of (α, β, γ) triple systems," Linear and Multilinear Algebras, **58** no.5-6 (2010) 617-643. This

history is a story without using concept of root systems and Cartan matrix in Lie algebras, in particular, is a study for triple systems.

Note that there are a lot of mathematician in nonassociative algebras (for Lie algebras), but a little groups in triple systems or Jordan algebras. For example, Profs. E. Zelmanov, K. McCrimmon, B. Allison, V. Kac, I. Shestakov, H. Petersson, M. Racine, H. Asano, I. Satake, M. C. Myung, A. Elduque, C. Martinez, S. Okubo and author, may be, only a few. Furthermore in addition, the book "*A Taste of Jordan Algebras*" (Springer, 2003) written by Prof. K. McCrimmon of a student in N. Jacobson is described about a history of the Jordan river. It here emphasize that this historical survey of certain Jordan algebras until the end of the 20th century and the beginning of 21th century is my (author) aspect (viewpoint). In addition to above river, for a certain example, for our imaginative illustrations with respect to a generalization of numbers;

$$\begin{aligned} \mathbf{R} \rightarrow \mathbf{C} \rightarrow \mathbf{H} \rightarrow \mathbf{O}(\text{octonion}) \rightarrow \mathbf{H}_3(\mathbf{O})(\text{Jordan algebra of 27 dim}) \rightarrow \\ \mathbf{M}(\mathbf{H}_3(\mathbf{O}))(\text{metasymplectic geometry of 56 dim}) \rightarrow \\ \mathbf{T}(\mathbf{H}_3(\mathbf{O}))(\text{symmetric space of 112 dim}) \rightarrow \\ \mathbf{E}_8(\text{exceptional simple Lie algebra of 248 dim}). \end{aligned}$$

On the other hand, there is other river also,

$$\begin{aligned} \mathbf{O} \rightarrow \mathbf{C} \otimes \mathbf{O}, \mathbf{H} \otimes \mathbf{O} \text{ and } \mathbf{O} \otimes \mathbf{O} (\text{Freudenthal's magic square}) \rightarrow \\ \mathbf{T}(\mathbf{O} \otimes \mathbf{O})(\text{symmetric space of 128 dim}) \rightarrow \mathbf{E}_8. \end{aligned}$$

It seems that there are several researchers group's tradition for these study and furthermore, for a nonassociative world of 21th century, Spanish, Portuguese and middle Europe scholars groups will glow with respect to the study.

For algebraic structures of nonassociative subject (AMS classification 17) related with geometry, we may describe as follows, for example (in my opinion);

Jordan algebras researchers (E. Artin origin),
Lie algebras researchers (N. Jacobson origin).

In summarizing about Jordan algebras or triple systems, we have the following diagrams (a generalization of complex and quaternionic numbers):

octonion, pseudo octonion algebras and triple systems $\implies$

Jordan algebras +Lie (super)algebras +symmetric composition algebras

$\implies$ **mathematical algebras**

In final comments (although they had described in the introduction), also we emphasize that nonassociative algebras are rich in algebraic structures, and they provide important common ground for various branches of mathematics, not only pure algebra and mathematical physics (for example, Pierce decompositions, Yang-Baxter equations and quark theory), but also analysis (Jordan C^* algebras or JB^* triple), topology (racks or quandles), and geometries (generalized symmetric spaces, convex cones or bounded symmetric domains, in particular). Hence, in future aspect, it seems that the triple systems (or ternary product) without using unit elements are useful concept for several subjects of sciences as well as the situation of symmetric spaces.

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These references are mainly papers for our study fields (as a historical survey article in these fields, so we have a lot of references).

Current address

Noriaki Kamiya
Chigasaki city Chigasaki 1-2-47-201, Japan
e-mail; shigekamiya@outlook.jp

ON PARALLELEPIPEDS IN ALGEBRA AND TOPOLOGY

Igor Bayak

Belarus, 230025, Grodno Av.
Kosmonavtov 4V 45
bayak@tut.by

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Abstract

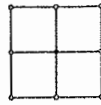
The concept of using parallelepipeds can be found in such branches of mathematics as combinatorial analysis, multilinear algebra and algebraic topology. Indeed, the sequences of edges of an n -dimensional parallelepiped stretched on the basis of an n -dimensional space are isomorphic to substitutions. In turn, the calculation of the oriented volume of a parallelepiped as a multilinear function taking a zero value on linearly dependent vectors leads to the concept of an alternating multilinear product. Finally, parallelepipeds can easily fold into cellular spaces and form chain complexes with a homology group as a topological invariant of these spaces. Bearing in mind the above-mentioned penetrations of geometry into algebra and topology, we will here consistently develop the algebra-topological formalism related to parallelepipeds, and then find its application in solving the problem of classification of closed orientable manifolds of arbitrary dimension.

MSC: 57Z05

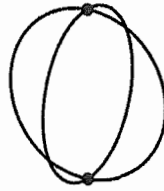
Keywords: combinatorial analysis, cellular spaces, minimal surfaces

1 Introduction

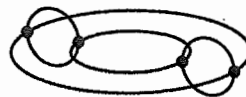
To quickly dive into the topic and acquire the skills of constructing frameworks of closed manifolds of arbitrary dimension, let us first turn to intuitive images of frameworks of 2-dimensional manifolds. First of all, let's take a 4-cell grid and



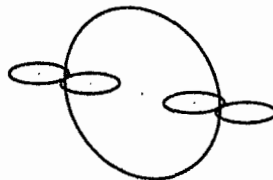
form the frameworks of the main 2-dimensional manifolds from it. In order to get the frame of the sphere, it is necessary to form a cross by moving the diagonal vertices (pull together with the edges) to the central vertex, and then identify the ends of the cross. The torus frame is obtained by identifying pairs of vertices lying

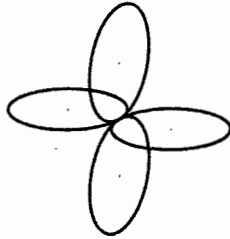


on two opposite horizontal and vertical sides of the grid. To form the frame of the



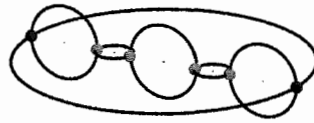
projective plane, it is necessary to identify pairs of vertices on opposite vertical sides and triples of vertices on horizontal sides of the grid. In turn, to form the



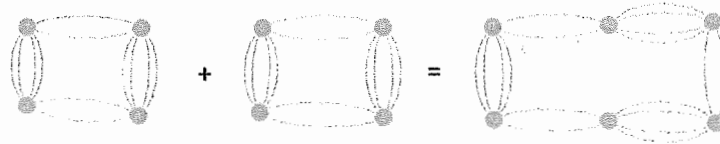


frame of the Klein bottle, it is necessary to identify the triples of vertices on both horizontal and vertical sides of the grid.

Next, we turn to the question of the classification of frameworks of 2-dimensional closed orientable manifolds. It is easy to notice that the pretzel frame is obtained by the interweaving of two torus frames. To do this, in each torus frame, it is necessary to uncouple the circles at one arbitrary vertex, and then identify the corresponding pairs of circles from two different frames. It is



clear that in the same way it is possible to obtain the framework of a 2-manifold having genus $g > 2$. However, it is more valuable that we can depict the frameworks of manifolds having a dimension of more than two. See, for example, how you can use graphs to depict the framework of a 3-manifold obtained by interweaving the frameworks of two 3-manifolds $S^2 \times S^1$. In the future (in the section



on parallelepipeds and manifolds), we will show how, using a cellular partition of a closed manifold, we can give a strict definition of the framework of a manifold and how to use a graph to depict the framework of a product of spheres of arbitrary dimension.

2 Permutations

First of all, let's turn to the foundations of combinatorial analysis, i.e. to the concept of mapping sets where the classical ones are established definitions, and therefore it will be sufficient for us to refer to a University textbook, for example, [2]. However, even classical representations sometimes allow for a certain modernization. Indeed, let be given such a triple of sets (A, B, F) that $F = \{(a, b)\} \neq \emptyset$, where $a \in A, b \in B$. Then we define the non-classical notion of mapping $F \subset A \times B$ the classical notation $f : A \rightarrow B$, meaning that each a pair of F can be represented by the matching arrow $a \rightarrow b$, where $b = f(a)$ for an unambiguous match and $b \in f(a)$ for a multi-valued correspondence. However, the triplet (B, A, F^{-1}) , where all arrows are reversed, sets the reverse display. If $\forall a \exists b (f(a) = b)$, then we are talking about unambiguous mapping of all A and thus let's return to the classic representation of the display. Narrowing the display of F to $A' \subset A$ we call this mapping $f' = f|_{A'} : A' \rightarrow B$, which corresponds to triplet (A', B, F') , where the inclusion of $F' \subset F$ is generated by including $A' \subset A$; i.e., excluding arrows from F , coming from a subset of $A \setminus A'$.

The expansion of the concept of display entails the expansion of such concepts such as injective and surjective mapping. Injective a mapping could be called F , which is its first the argument exhausts A , and in which there are no converging arrows, i.e. there are no pairs $(x, b), (y, b)$, in other words, $\forall x \neq y \in A \rightarrow f(x) \cap f(y) = \emptyset$. In turn, a surjective map could be called F , which its second argument exhausts B , and in which there is no divergent arrows, i.e. there are no pairs $(a, x), (a, y)$, otherwise in other words, $\forall b \exists a (f(a) = b)$. A quasi-objective map can be called F , which exhausts A , and in which there are no diverging arrows, otherwise in other words, $\forall a \exists b (f(a) = b)$. A quasi-subjective map can be called F , which exhausts B , and in which there are no converging arrows, in other words, $\forall b \exists! a (f(a) = b)$. Finally, strictly an injective map can be called a map that is injective in in the classical sense, i.e. such that $\forall x \neq y \in A \Rightarrow f(x) \neq f(y)$. Similarly, strictly surjective a mapping can be called a mapping that is surjective in in the classical sense. However, in the concept of bijective but we still put a classic meaning in it.

The logical relationships between all these types of mappings are reflected in the following obvious statements.

2.1 Proposition. *Bijective mapping is both injective and surjective*

2.2 Proposition. *The narrowing of the bijection is strictly injective*

2.3 Proposition. *The inverse of an injective (quasi-injective) map is a surjection (quasi-surjection)*

Now, within the framework of the above formalism, it will be easy for us determine the main combinatorial concepts. Indeed, if an arbitrary bijection from the triplet (A, A, F) is called a substitution, and the simplest substitution is — transposition, then placement without repetitions can be defined as a strict injection, and placement with repetitions — as a quasi-injection. We will also take a convenient measure notation. If sets $I = \{1, \dots, n\}$, $J = \{1, \dots, m\}$ are specified, where $m \leq n$, then the mapping $/J/ := /i_1, \dots, i_j, \dots, i_m/ : J \rightarrow J$ let denotes an arbitrary substitution of m elements of the set J , and $\text{sgn } /J/$ — the parity sign of this substitution; display $[J] := [i_1, \dots, i_j, \dots, i_m] : J \rightarrow I$ let denotes an arbitrary placement from n to m with repetition. Display $\langle J \rangle := \langle i_1, \dots, i_j, \dots, i_m \rangle : J \rightarrow I$ denotes an arbitrary placement from n to m without repetitions; display $(J) := (i_1, \dots, i_j, \dots, i_m) : J \rightarrow I$ let denote arbitrary ordered placement, i.e. such $\langle J \rangle$ in which $i_j < i_{j+1}$. Finally, if the set $\{J\} := \{i_1, \dots, i_j, \dots, i_m\}$ denotes us derived from (J) , a subset of I consisting of m elements for placing (J) , then mapping $/(J)/ := /(i_1, \dots, i_j, \dots, i_m)/ : \{J\} \rightarrow \{J\}$ let denotes an arbitrary substitution of elements of the set $\{J\}$, accordingly, $\text{sgn } /(J)/$ — the parity sign of this substitution.

Let the mapping

$$/(J)\hat{i}_j/ : (i_1, \dots, i_j, \dots, i_m) \rightarrow (i_1, \dots, \hat{i}_j, \dots, i_m)i_j$$

denotes the derivative of (J) and an arbitrary element $\{J\}$ is a substitution for removing this element, as a result of which the element i_j goes to the end and the remaining elements, after transferring i_j , remain ordered in ascending order. Along with removing an element from an ordered set, you can consider attaching an element with subsequent ordering. Then we get the substitution $/(J)\check{k}/$, where $k \in \{J'\} = I \setminus \{J\}$, which should be understood as a substitution of the ordering of the set $\{(J), k\}$. All these substitutions have the anti-commutativity property of their parity values, so that $\text{sgn } /(J)\hat{i}_j\hat{i}_k/ = -\text{sgn } /(J)\hat{i}_k\hat{i}_j/$, and $\text{sgn } /(J)\check{j}\check{k}/ = -\text{sgn } /(J)\check{k}\check{j}/$.

Finally, we consider the partial substitution violations orderliness of the set I . Assume that the substitution image $/J, J'/ : I \rightarrow \{(J), (J')\}$, where $\{J'\} = I \setminus \{J\}$, formed by the sets $\{J\}$ and $\{J'\}$, ordered by separately, and the substitution $/(J, J')/ : \{(J), (J')\} \rightarrow \{(J'), (J)\}$ implies a permutation of ordered sets. Then, since $/J', J/ = /J, J'/ * /(J, J')/$ and $\text{sgn } /(J, J')/ = (-1)^{m(n-m)}$, then we get the equality $\text{sgn } /J, J'/ \cdot \text{sgn } /J', J/ = (-1)^{m(n-m)}$.

3 Alternating products

The algebra of alternating multilinear products is well studied, and so you can refer, for example, to the corresponding Chapter in [4] or [9]. Here we are using the concept of parity partially ordered substitutions and elimination substitutions (2), we will touch only on certain issues.

First of all, let's turn to multilinear products in general. Let $\mathbb{L}$ means n -dimensional linear space over $\mathbb{R}$ and $(e^i)_I$ — its basis. Let also $\mathbb{T}^m(\mathbb{L})$ means n^m -dimensional space over $\mathbb{R}$ and $(e^{[J]})_{\{[J]\}}$ — its basis. Then we can explicitly represent a tensor multilinear product if write

$$x^1 \otimes \dots \otimes x^m : \mathbb{L}^m \rightarrow \mathbb{T}^m(\mathbb{L}) : (x^1, \dots, x^m) \rightarrow \sum_{\{[J]\}} \prod_{i_j \in [J]} x_{ij} e^{[J]}$$

where x_{ij} means j -th coordinate of the vector x^i . Because $e^{[i_1]} \otimes \dots \otimes e^{[i_m]} = e^{[J]}$, then linear span of the set $\{e^{[i_1]} \otimes \dots \otimes e^{[i_m]}\}_{\{[J]\}}$ isomorphic to a tensor linear space $\mathbb{T}^m(\mathbb{L})$.

Now let's form two alternating multilinear products, and exactly, the outer product of

$$x^1 \wedge \dots \wedge x^m : \mathbb{L}^m \rightarrow \Lambda^m(\mathbb{L}) : (x^1, \dots, x^m) \rightarrow \sum_{\{(J)\}} e^{(J)} \sum_{\{/(J)/\}} \text{sgn } /(J)/ \prod_{i_j \in /(J)/} x_{ij} = \sum_{\{(J)\}} \det(x_{ij})_J^{(J)} e^{(J)}$$

where $\Lambda^m(\mathbb{L})$ is a polyvector space, isomorphic to the linear shell of a set $\{e^{(i_1} \wedge \dots \wedge e^{i_m})\}_{\{(J)\}}$; and vector product

$$x^1 \times \dots \times x^m : \mathbb{L}^m \rightarrow \Lambda^{n-m}(\mathbb{L}) : (x^1, \dots, x^m) \rightarrow \sum_{\{(J')\}} \text{sgn } /J, J'/ \det(x_{ij})_J^{(J)} e^{(J')},$$

where $\{J'\} = I \setminus \{J\}$. All these works are guaranteed alternation, since the transposition of an arbitrary pair vectors from the set $(x^1, \dots, x^m)$ parity signs of all the $/ (J) /$ substitutions are reversed. In addition, for that's fair

3.1 Proposition. *The vector system $(x^1, \dots, x^m)$ is linearly dependent if and only if $x^1 \wedge \dots \wedge x^m = x^1 \times \dots \times x^m = 0$*

Indeed, on the one hand, if the system $(x^1, \dots, x^m)$ is linearly dependent, then $\det(x_{ij})_J^{\{J\}} = 0 \quad \forall (J) \in \{(J)\}$, on the other hand, if the system $(x^1, \dots, x^m)$ is linearly independent, then using elementary transformations of the system and permutations of elements basis $(e^1, \dots, e^n)$ matrix $(x_{ij})_J^I$ is reduced to a trapezoidal form with m nonzero strings and nonzero diagonal $(x_{ii})_J$, which is why $\det(x_{ij})_J^J \neq 0$. Here we took advantage of the the fact that the elementary transformations preserve independence of the vector system and at the same time maintain equality (inequality) to zero of the determinant, and substitutions of the basis do not take place neither external nor vector product. Thus, the external and the vector product of a linearly independent system of vectors has at least one non-zero coordinate, and therefore the statement is proved.

Polyvector space $\Lambda^m(\mathbb{L})$ interesting yet and the fact that there are natural lowering and increasing boundary homomorphisms, namely,

$$\delta : \Lambda^m(\mathbb{L}) \rightarrow \Lambda^{m-1}(\mathbb{L}) : e^{(J)} \rightarrow \sum_{j \in \{J\}} \text{sgn} / (J) \hat{j} / \cdot e^{(\check{J})},$$

where $(\hat{J})$ means an ordered substitution of an exception $/ (J) \hat{j} /$, and

$$d : \Lambda^m(\mathbb{L}) \rightarrow \Lambda^{m+1}(\mathbb{L}) : e^{(J)} \rightarrow \sum_{j \in I \setminus \{J\}} \text{sgn} / (J) \check{j} / \cdot e^{(\check{J})},$$

where $(\check{J})$ means an ordered substitution of $/ (J) \check{j} /$. Indeed, since in the polyvector $\delta\delta(x)$ basic elements are formed using exceptions twice, then by virtue of the property anticommutativity of parity substitutions with an exception we get identity $\delta\delta(x) = 0$ for any polyvector x . Similarly, the identity $dd(x) = 0$ is obtained.

Let's now turn to the space $\mathbb{L}$, in which you specify the functional scalar product. Let $\mathbb{E}$ be an arbitrary n -dimensional Euclidean space over $\mathbb{R}$. If we assume that

$$x^1 \wedge \dots \wedge x^m \cdot y^1 \wedge \dots \wedge y^m = \det(x^i \cdot y^j)_J,$$

then this means that we induce from $\mathbb{E}$ to $\Lambda^m(\mathbb{E})$ scalar product of polyvectors. If you take the square of the polyvector $x^1 \wedge \dots \wedge x^m$, i.e. its the scalar product on itself, then we get the determinant Gram of the vector system $(x^1, \dots, x^m)$, equal to zero which is equivalent to the linear dependence of this system. Linearization of the gram determinant, i.e. the square root of it absolute value, is a polyline (but not alternating) function, which is usually identified with the volume a parallelepiped stretched over the system $(x^1, \dots, x^m)$. Recall that the scalar product

establishes an isomorphism between $\mathbb{E}$ and its dual space $\mathbb{E}^*$ with the basis $(\varepsilon^i)_I$, dual to the orthonormal basis $(e^i)_I$, i.e.

$$* : \mathbb{E} \rightarrow \mathbb{E}^* : x \rightarrow x^* : \langle x^*, y \rangle = x \cdot y \quad (\forall x, y \in \mathbb{E}) :$$

$$\langle e_i^*, e_i \rangle = e_i \cdot e_i : e_i^* = e_i \cdot e_i \varepsilon_i,$$

which induces an isomorphism $\mathbb{T}^m(\mathbb{E}) \simeq \mathbb{T}^m(\mathbb{E}^*)$. In particular, the isomorphism $\Lambda^m(\mathbb{E}) \simeq \Lambda^m(\mathbb{E}^*)$ set by the correspondence between m-vectors and m - forms by formula

$$* : \sum x_J e^J \rightarrow \sum x_J^* \varepsilon^J : \sum_{\{J\}} x_{(J)} e^{(i_1 \wedge \dots \wedge e^{i_m})} \rightarrow \sum_{\{J\}} x_{(J)} e^{(*i_1 \wedge \dots \wedge e^{*i_m})}$$

We also recall that the Euclidean structure of a linear space sets the operator the Hodge dualization

$$\star : \Lambda^m(\mathbb{E}) \rightarrow \Lambda^{n-m}(\mathbb{E}) :$$

$$\sum x_J e^J \rightarrow \langle * \sum x_J e^J, \epsilon \rangle : \sum x_J e^J \rightarrow \text{sgn } /J, J' / \sum x_J^* e^{J'},$$

where is the discriminant $\epsilon := e^1 \wedge \dots \wedge e^n$ when collapsing from m-the form e^J acquires the wildcard character $e^J \wedge e^{J'} := e^{(i_1 \wedge \dots \wedge e^{i_m})} \wedge e^{(i_{m+1} \wedge \dots \wedge e^{i_n})}$. However, in Euclidean a space with a positive signature, where $x_J^* = x_J$, we get $\star : \Lambda^m(\mathbb{E}) \rightarrow \Lambda^{n-m}(\mathbb{E}) : \sum x_J e^J \rightarrow \text{sgn } /J, J' / \sum x_J e^{J'}$. Thus, our definition of a vector works using the parity of permutations of partially ordered consistent with the definition of the dualization operator in Euclidean space with a positive signature through the fold with the discriminant.

Let us now turn to smooth differential forms. Let it be given field of smooth differential m-forms $a(x) = \sum a_J(x) dx^J$, where $J := (J)$, $dx^J := dx^{(i_1 \wedge \dots \wedge dx^{i_m})}$ and $a_J(x)$ — arbitrary smooth mappings from $\mathbb{L}$ in $\mathbb{R}$. Recall that the external product differential m-form $a(x) = \sum a_J(x) dx^J$ and the differential 1-form $b(x) = \sum_n b_i(x) dx^i$ is called ((m+1)-form $a(x) \wedge b(x) = \sum b_i(x) a_J(x) \text{sgn } \check{J} dx^{\check{J}}$, where $\check{J}$ is an ordered substitution joins $/ (J) i /$, i.e. the outer product is obtained in the result of the external product of these forms and substitutions the annexation that transforms the basis elements $dx^J \wedge dx^i$ to the ordered form $dx^{\check{J}}$.

However, in addition to the external product, there is an internal one product of the differential m-form $a(x) = \sum a_J(x) dx^J$ on the vector field $\bar{b}(x) = \sum_n b_i \frac{\partial}{\partial x_i}$,

which refers to the convolution of a form field with vector field. However, despite the simplicity of this definition for an internal work, we draw your attention to the fact that for collapsing must be performed by substituting the withdrawal, resulting in the form $a(x) = \sum a_J(x) dx^J$ to the form $\sum a_J(x) \text{sgn } \hat{J} dx^{\hat{J}} \wedge dx^i$, and then replacing in this view an external product on dx^i on a product on number $b_i(x)$, get the inner product you are already looking for $\langle a(x), \bar{b}(x) \rangle = \sum b_i(x) a_J(x) \text{sgn } \hat{J} dx^{\hat{J}}$.

External derivative of a smooth differential form (differential) given by the formula $da(x) = \sum d_J a(x) dx^J = \sum da_J(x) \wedge dx^J$, where $da_J(x) = \sum_n \frac{\partial a_J(x)}{\partial x_i} dx^i$. Then, in force properties of anti-commutativity of ordering substitutions with joining that occur when re-differentiating, we get the identity $da(x) = 0$, which characterizes the external the derivative increases as the boundary homomorphism.

Now let's set another important property of the derivative's appearance. Let be given such an arbitrary extremely small vector $\Delta x = \sum_n \Delta x^i = \sum_p \Delta x_i e^i$, what can be done from it generate a non-zero $(m+1)$ -vector $\Delta = \sum \Delta x^J$, where $\Delta x^J := \Delta x^{(i_1 \wedge \dots \wedge \Delta x^{i_{m+1}})}$, and the corresponding m is the vector $\delta \Delta = \sum \Delta x^J$, where $\Delta x^J = \Delta x^{(i_1 \wedge \dots \wedge \Delta x^{i_m})}$. Then

$$\langle da(x), \Delta \rangle = \sum \langle d_J a(x) dx^J, \Delta x^J \rangle$$

where $\langle d_J a(x) dx^J, \Delta x^J \rangle = d_J a(x) \cdot \Delta x_{i_1} \dots \Delta x_{i_{m+1}}$. On the other hand, decomposing in this expression the differential component and the polyvector component, we will have expression $\langle d_J a(x) dx^J, \Delta x^J \rangle = \langle \sum_n \frac{\partial a_J(x)}{\partial x_i} dx^i \wedge dx^J, \sum_n \Delta x^i \wedge \Delta x^J \rangle$, and if then collapse 1-form and vector, then we get the equality $\langle d_J a(x) dx^J, \Delta x^J \rangle = \langle \sum_{m+1} [a_J(x + \Delta x^i) - a_J(x)] dx^J, \sum_{m+1} \Delta x^J \rangle$. Thus, it is proved statement

3.2 Proposition. $\langle da(x), \Delta \rangle = \langle [a(x + \Delta x) - a(x)], \delta \Delta \rangle$

We will apply this offer in the next section when making a withdrawal Stokes formulas, where extremely small vectors serve as the backbone for extremely small parallelepiped a integration over the surface is identified with summation of convolutions of differential form and of multivectors corresponding to those of a parallelepiped. At once, note only that the value $\langle da(x), \Delta \rangle$ characterizes the linear part of the convolution increment of the differential the shape at the point x and the polyvector corresponding to the faces parallelepipeds constructed on extremely small vectors, when shifting each face to the opposite side and changing it accordingly values of the differential form.

4 Parallelepipeds and manifolds

We begin with the classical definition of parallelepipeds, which is here we repeat, referring to the University textbook [5].

4.1 Definition. *Parallelepiped π^m stretched linearly on the system independent vectors $x^1, \dots, x^m$ coming from a point x^0 n -dimensional affine space $\mathbb{R}^n$, the set of points is called $x^0 + \sum_m \alpha_i x^i$, set by the condition $0 \leq \alpha_i \leq 1 \quad \forall i \in J$.*

Then, in addition to this geometric concept, we will give algebraic definition of an oriented parallelepiped.

4.2 Definition. *An oriented parallelepiped is called a parallelepiped π^m , in which the direction of all edge chains $x^{i_1}, \dots, x^{i_m}$ is specified in a certain way. The direction of the chain from x^0 to $x^0 + \sum_m x^i$ is considered positive, and the reverse is considered negative. If in a parallelepiped all even chains (chains with even sequences of edges) are given a positive direction, and odd ones are given a negative direction, then such a parallelepiped is called positively oriented and is denoted $+\pi^m$, i.e. $\text{sgn } \pi^m = +1$. In the opposite case, the parallelepiped is called negatively oriented and is denoted $-\pi^m$, i.e. $\text{sgn } \pi^m = -1$.*

Based on the geometric definition, we assume that the faces parallelepiped π^m serve $2m$ parallelepiped π^{m-1} , namely, m of parallelepipeds π_{*j}^{m-1} , vectors stretched on subsystems $m-1$ obtained by the exception vectors x^j , where $j \in J$, from the system $x^1, \dots, x^m$, and from the starting point x^0 , as well as m parallelepipeds π_{j*}^{m-1} coming from m points $x^0 + x^i$ and stretched respectively on subsystems without x^j each. Thus, y parallelepiped 2^m vertices, $2m$ faces, and $m2^{m-1}$ edges. Note also that all faces of parallelepiped faces are π^m connect the corresponding pairs of adjacent faces, i.e.

$$\pi_{j*}^{m-1} \cap \pi_{k*}^{m-1} = \pi_{jk*}^{m-2}, \quad \pi_{*j}^{m-1} \cap \pi_{*k}^{m-1} = \pi_{*jk}^{m-2},$$

$$\pi_{j*}^{m-1} \cap \pi_{*k}^{m-1} = \pi_{j*k}^{m-2}$$

where $j \neq k$.

4.3 Definition. *The oriented surface of a parallelepiped is called a set of faces oriented in a certain way. The orientation of the π_{*j}^{m-1} faces of the parallelepiped corresponds to parity substitutions $/(1, \dots, m)\hat{j}/$ a faces π_{j*}^{m-1} that are opposite to the faces of π_{*j}^{m-1} have also the opposite orientation. In other words, $\text{sgn } \pi_{*j}^{m-1} = -\text{sgn } \pi_{j*}^{m-1} = \text{sgn } /(1, \dots, m)\hat{j}/$.*

Then, due to the anticommutativity of substitutions with withdrawal, we get one remarkable property

4.1 Proposition. *Common faces of adjacent faces of an oriented surface parallelepipeds have the opposite orientation in them.*

Indeed, in the parallelepiped π_{*j}^{m-1} , the orientation of the face is π_{*jk}^{m-2} equals $\text{sgn} / (1, \dots, m) \hat{j} \hat{k} /$ and in the parallelepiped π_{*k}^{m-1} the orientation of the same face is equal to

$$\text{sgn} / (1, \dots, m) \hat{k} \hat{j} /$$

Similarly, in the parallelepiped π_{j*}^{m-1} the orientation of the face π_{jk*}^{m-2} is equal to

$$\text{sgn} / (1, \dots, m) \hat{j} \hat{k} /$$

and in the parallelepiped π_{k*}^{m-1} its orientation is

$$\text{sgn} / (1, \dots, m) \hat{k} \hat{j} /$$

However, in the parallelepiped π_{j*}^{m-1} the orientation of the face π_{j*k}^{m-2} is $-\text{sgn} / (1, \dots, m) \hat{j} \hat{k} /$ and in the parallelepiped π_{*k}^{m-1} the orientation is $-\text{sgn} / (1, \dots, m) \hat{k} \hat{j} /$. Algebraic construction of an oriented parallelepiped allows for formalization. Indeed, if we assume multiplicity oriented parallelepiped implemented by bundles of chains of its edges, then an algebraic sum of numbers positive and negative oriented layers can be set some arbitrary integer. Thus, we can speak about the-oriented parallelepiped $z\pi^m$, which is an element of the Abelian group Π_m , isomorphic to $\mathbb{Z}$.

In turn, the geometric construction of a parallelepiped through possibility of forming a homological network of parallelepipeds allows an extension to the concept of *cell surface*. Indeed, a parallelepiped π^m in space $\mathbb{R}^n$ can be continued geometrically, via proper $(m-1)$ -dimensional faces so that the new m -dimensional the parallelepiped is stretched on an arbitrary, but already existing one a face, and some new vector that does not lie in the subspace of this facets. Finite or countable union of parallelepipeds π^m , whose arbitrary pairs either do not intersect at all, or intersect with their common faces of arbitrary dimension, it can be identified with *m-dimensional cell complex*. Then *m-dimensional cell surface* Ω^m we call such a m -dimensional cell complex in which each pair of parallelepipeds can be connected to a chain connected by the common $(m-1)$ -dimensional faces of its links – neighboring parallelepipeds. However, *m-dimensional cell manifold*

we call a cell surface without self-intersections, i.e. one in which each common face of the surface has dimension $(m - 1)$ and is the intersection of only two parallelepipeds. *Boundary* $d\Omega^m$ of a cell manifold Ω^m we call the union of all $(m-1)$ -dimensional faces of parallelepipeds of the manifold Ω^m minus the common faces. *Closed cell manifold* we call a compact (composed of a finite number of parallelepipeds) cell manifold without borders. Note that among all the faces of a single parallelepiped π^m , there is no common face, so the cell surface of the parallelepiped is its boundary $d\pi^m$. At the same time, since all $(m - 2)$ -dimensional faces of the surface of the parallelepiped $d\pi^m$ are common faces, the surface of the parallelepiped has no boundary, and therefore $dd\pi^m = \emptyset$. However, since every cell manifold is a union of parallelepipeds, then for every cell manifold Ω^m the identity $dd\Omega^m = \emptyset$ is valid, and therefore the boundary of the manifold $d\Omega^m$ is either a closed manifold, or it is a union of closed manifolds.

An elementary homeomorphic transformation of a cell manifold with a boundary Ω^m we call a connection to the boundary of a new parallelepiped or a removal of an existing parallelepiped that does not violate the connectedness of its boundary, that is, does not change the number of pieces (closed manifolds) of its boundary. In turn, an elementary homeomorphic transformation of a closed cell manifold $d\Omega^m$ we mean an elementary homeomorphic transformation of a cell manifold with boundary Ω^m .

The set of all m -dimensional parallelepipeds that make up the manifold Ω^m generates a free Abelian group $\Pi_m(\Omega^m)$ formed by the formal sums $\sum_K z_i \pi_i^m$, where $\sum_K z_i \pi_i^m$, $\forall z_i \in \mathbb{Z}$, $K \subseteq \mathbb{N}$. The sum $\sum_K \pm \pi_i^m$, composed of identically oriented parallelepipeds of the manifold Ω^m , whose intersection faces of neighboring parallelepipeds have the opposite orientation in them, we call oriented cell manifold. A cell manifold that allows the orientation of its parallelepipeds is called *orientable cell manifold*. We note in this case that with an elementary homeomorphic attachment of a parallelepiped, the face of the connection of this parallelepiped with a parallelepiped adjacent to it has the opposite orientation in them only if the parallelepipeds themselves have the same orientation. Therefore, the orientation of the homeomorphically attached (withdrawn) parallelepiped is consistent with the orientation of neighboring parallelepipeds, and therefore we obtain

4.2 Proposition. *Orientability is a topological invariant of the cell manifold Ω^m*

In addition, every cell manifold Ω^m generates a free Abelian group $\Pi_{m-1}(\Omega^m)$ formed by formal sums of the faces of its parallelepipeds. Group $\Pi_{m-1}(\Omega^m)$ we

call the face group of a cell variety. The group of faces of a single parallelepiped $\Pi_{m-1}(\pi^m)$ consists of the formal sums $\sum_{2m} z_i \pi_i^{m-1}$ and the mapping

$$\gamma : \pi^m \rightarrow \Pi_{m-1}(\pi^m) :$$

$$\pi^m \rightarrow \sum_m \text{sgn} / (1, \dots, m) \hat{j} / \cdot \pi_{*j}^{m-1} - \sum_m \text{sgn} / (1, \dots, m) \hat{j} / \cdot \pi_{j*}^{m-1}$$

induces a boundary homomorphism of the corresponding Abelian groups

$$\delta : \Pi_m(\Omega^m) \rightarrow \Pi_{m-1}(\Omega^m) : \sum_K z_i \pi_i^m \rightarrow \sum_K z_i \gamma(\pi_i^m).$$

Valid, by virtue of the offer 4.1 has the identity $\gamma\gamma(\pi^m) = 0$, applying which is component-wise, we get another identity $\delta\delta(\Omega^m) = 0$, defining the boundary map and chain complex $\{\Pi_{m-1}(\Omega^m), \delta\}$.

Put

$$Z_{m-1}(\Omega^m) := \text{Ker}\{\delta : \Pi_{m-1}(\Omega^m) \rightarrow \Pi_{m-2}(\Omega^m)\} \subset \Pi_{m-1}(\Omega^m)$$

and let's call $Z_{m-1}(\Omega^m)$ a group of cycles. Suppose also

$$B_{m-1}(\omega^m) := \text{Im}\{\Delta : \Pi_m(\omega^m) \rightarrow \Pi_{m-1}(\Omega^m)\} \subset \Pi_{m-1}(\Omega^m)$$

and let's call $B_{m-1}(\Omega^m)$ a boundary group. Since $\delta\delta = 0$, then $B_{m-1}(\Omega^m) \subset Z_{m-1}(\Omega^m)$. Factor group

$$H_{m-1}(\Omega^m) := Z_{m-1}(\Omega^m) / B_{m-1}(\Omega^m)$$

let's call $(m-1)$ - dimensional homology group. Then, since the homeomorphic addition or removing a parallelepiped adds faces to the group or removes from it an elementary loop that is a boundary, then we have

4.3 Proposition. *The group of $(m-1)$ -dimensional homologies is topological invariant of the cell manifold Ω^m*

Thus, the application of the concept of an oriented parallelepiped allows you to make a natural transition to building a standard homology theory. However, we will show that parallelepipeds can also be successfully applied in the classification of closed cell manifolds.

Let the frame of a closed m -dimensional cell manifold be the graph obtained as a result of coloring the m -dimensional manifold 2^m with paint colors. If there

are not enough parallelepipeds for coloring a manifold, then they are added by a homeomorphic join. The coloring begins with a bundle of 2^m parallelepipeds assembled at a single intersection point, which are then colored in different colors. Next, the parallelepipeds are colored in any color, but with such a condition that the color of the colored parallelepiped coincides with the color of one of the neighboring ones. *frame vertices* is formed as the intersection points of 2^m parallelepipeds of different colors, and *frame edges* are formed as a result of the intersection of 2^{m-1} parallelepipeds of different colors. For reasons of symmetry, it follows that the frame of a m -dimensional closed cell manifold can have any number of vertices, but each vertex must have a set of m pairs of edges. In this case, if none of these pairs of edges forms a pair of loops, then this is the frame of an oriented cell manifold, and if at least one of these pairs of edges forms a pair of loops, then this will be the frame of an undirected closed cell manifold.

Let us now consider the question of classifying the frames of orientable closed cell manifolds, which for brevity we will call *c-frames*. If we replace pairs of *c*-edges of a n -dimensional manifold with single edges, we see that the classification of *c*-frames is equivalent to the classification of undirected graphs without loops, which have the property that each vertex of the graph is incident to n adjacent edges.

Now let's form an elementary *c*-frame corresponding to the simplest graph. To do this, we arbitrarily decompose n into the sum of $n_1 + \dots + n_m = n$, where $1 \leq m \leq n$, and build 2^m -vertex graph isomorphic to m -parallelepiped with edges of multiplicity n_i , i.e. such a graph whose vertices coincide with the vertices of m -parallelepiped and all n_i of its multiples ribs correspond to each edge of the parallelepiped from the corresponding the class of its parallel edges. Then, since m is a parallelepiped of $m2^{m-1}$ edges, taking into account their multiplicity, we get $n2^{m-1}$ edges of the graph or $n2^{m-1}$ pairs of edges of the corresponding *c*-frame. Elementary *c*-frame, corresponding to the decomposition of $n_1, \dots, n_m$ we denote with the symbol $S^{n_1} \times \dots \times S^{n_m}$ and note that two arbitrary elementary *c*-frames can be composed new *c*-frame $S^{n_1} \times \dots \times S^{n_m} + S^{n_1} \times \dots \times S^{n_l}$, which corresponds to a graph obtained from two simplest graphs by breaking edges in one arbitrary vertex of each of them and the next one combining the corresponding edges of these two graphs. However, if $l = 1$, then we get an exception, since in this case $S^{n_1} \times \dots \times S^{n_m} + S^n = S^{n_1} \times \dots \times S^{n_m}$.

So, we have a complete classification of *c*-frames, which is determined by their Assembly from elementary *c*-frames. Note also that a cell variety that has *c*-frame of elementary type $S^{n_1} \times \dots \times S^{n_m}$ is homeomorphic to the product of

spheres $S^{n_1} \times \dots \times S^{n_m}$. In turn, a variety with c -frame represented by the sum of two elementary c -frames is formed from the corresponding products of spheres using a connected sum, i.e. by cutting out a cell in each of them and gluing them along the cell boundary. In view of these comments, the statement is valid.

4.4 Proposition. *The fundamental group of a closed orientable manifold trivial if in all its forming elementary frames of the type $S^{n_1} \times \dots \times S^{n_m}$ is not found in any one components of S^1*

Proof. Indeed, under these conditions, the original variety can be glue from elementary manifolds, each of which can be decompose into the product of submanifolds that are homeomorphic to some spheres S^k , where each $k > 1$. But since the fundamental the product group of such spheres is trivial and each border is glued is homeomorphic to the sphere S^{n-1} , which in the case of $n > 2$ has a trivial fundamental group, then the statement is proved $\square$

From the last sentence and from the fact that the classification of closed orientable manifolds in dimension three are reduced to classifications of corresponding wireframes that can be compiled from frames of type $S^1 \times S^1 \times S^1$, $S^2 \times S^1$ and $S^1 \times S^2$, where it is found component S^1 , or it is a frame of type S^3 , implies that closed oriented 3-manifolds with trivial fundamental group homeomorphic to a 3-dimensional sphere. Thus, the classical Poincare hypothesis is a simple consequence of classifying closed orientable manifolds by classifying their frames.

Let us now turn to measurement issues related to cell surfaces. Because the outer product of the vectors that the parallelepiped is stretched over sets the mapping $\bar{\pi}^m : \pi^m \rightarrow \wedge^m(\mathbb{L})$, inducing equivalence of space elements $\{\pi^m\}$, the space $\wedge^m(\mathbb{L})$ and $\wedge^m(\mathbb{L}^*)$ can be interpreted as space of equivalent devices and their effects respectively. In other words, we assume that the convolution $\langle \bar{x}, \tilde{y} \rangle$, where $\bar{x} \in \wedge^m(\mathbb{L})$, $\tilde{y} \in \wedge^m(\mathbb{L}^*)$, is the result of the impact measurement $\tilde{y}$ on the device $\bar{x}$. In accordance with this concept, every manifold Ω^m and the field differential m -forms $a(x)$ can be mapped to the sum of convolutions $\sum_K \langle a(x_i^0), \bar{\pi}_i^m \rangle$, which it should be understood as the result of exposure to the surface, composed of K devices–parallelepipeds. In the event, when the manifold Ω^m is composed of extremely small parallelepiped $\Delta\pi^m$, we will call it integral surface a the sum of the convolution is denoted by the integral symbol $\int_{\Omega^m} \langle a(x), \Delta\bar{\pi}^m \rangle$. Note that for a canonical surface, all parallelepipeds of which are built on vectors collinear with the basic ones, the designation is accepted $\int_{\Omega^m} a(x) dx^J$. Because common faces are both initial and final generating faces in adjacent parallelepipeds, then

applying decomposition

$$\langle da(x), \Delta \bar{\pi}^m \rangle = \sum_m \langle a(x + \Delta x^j), \Delta \bar{\pi}_{j*}^{m-1} \rangle - \sum_m \langle a(x), \Delta \bar{\pi}_{*j}^{m-1} \rangle$$

to all extremely small parallelepipeds of the integral surface, we get the formula Stokes

$$\int_{\Omega^m} \langle da(x), \Delta \bar{\pi}^m \rangle = \int_{\delta \Omega^m} \langle a(x), \Delta \bar{\pi}^{m-1} \rangle.$$

If the mapping $\bar{\pi}^m$ is induced by a vector product, then with the oriented manifold Ω^m you can link the vector volume

$$\text{Vol}(\Omega^m) := \sum_K \bar{\pi}_i^m$$

Finally, with every integral a surface defined in Euclidean space as positive signatures, you can link a linear volume

$$\int_{\Omega^m} \sqrt{|\Delta \bar{\pi}^m \cdot \Delta \bar{\pi}^m|}$$

which is in the observed space associated with length, area, and actual volume.

5 Minimal flows and surfaces

Vector fields and minimal surfaces defined in Euclidean space, thoroughly described in the educational literature [1] [6], [7], [8], so we'll limit ourselves to by considering only what directly follows from our parallelepiped algebras, or let's look at the known facts a little on the other side.

Let $\mathbb{R}^n$ be set in Euclidean space an arbitrary smooth covector field, i.e. a field of differential 1-forms $a(x) = \sum_n a_j(x) dx^j$. Then we have also conjugate to $a(x)$ vector field $a^*(x)$ and dual to $a(x)$ differential (n-1)-form $\star a(x) = \text{sgn } J \sum_n a_j(x) dx^J$, where $J := /I, \hat{j}/$. Differential the forms $a(x)$ and $\star a(x)$ can be represented as a stationary flow of a certain ideal liquid, measured by 1-dimensional and (n-1)-dimensional devices (polyvectors) as a result of their collapse from $a(x)$ and $\star a(x)$ respectively. If as devices for flow measurements are used 1-dimensional or (n-1)-dimensional integral surfaces, then as a result of the measurement we get integral sums $\int_{\Omega^1} \langle a(x), dx \rangle$, $\int_{\Omega^{n-1}} \langle \star a(x), dx^{n-1} \rangle$ accordingly, where $dx := \Delta \bar{\pi}^1$ and $dx^{n-1} := \Delta \bar{\pi}^{n-1}$. Application of extremely small surfaces

without borders gives us the ability to set local flow characteristic. Indeed, the lack of local rotation of the flow $\oint \langle a(x) dx^1 \rangle = 0$ provided the differential condition $da(x) = 0$, and the absence of drains and sources $\oint \langle \star a(x), dx^{n-1} \rangle = 0$ provided by the condition $d \star a(x) = 0$.

We will add criteria for the flow closure to its criteria stabilities. Thus, the laminar flow current characterized by its by lamination, satisfies the differential holonomy condition $a(x) \wedge da(x) = 0$. Note that every exact 1-form $a(x) = d\varphi(x)$ holonomic and its family of integrals surfaces (lamination) is given by the level surfaces of the function $\varphi(x)$. In turn, any holonomic covector field $a(x)$ collinear to some potential covector field $d\varphi(x)$, i.e. $a(x) = k(x)d\varphi(x)$, where $k(x)$ — an arbitrary non-zero smooth function. Really, $da(x) = d[k(x)d\varphi(x)] = dk(x) \wedge d\varphi(x)$, and so $a(x) \wedge da(x) = k(x)d\varphi(x) \wedge dk(x) \wedge d\varphi(x) = 0$. Conversely, if the equation $a(x) \wedge da(x) = 0$ is completely integrable, then its integral surfaces, i.e. the foliation, can always be put in correspondence with the level surface of some function $\varphi(x)$, and hence $a(x)$ collinear $d\varphi(x)$.

The equilibrium current of the laminar flow satisfies the corresponding differential condition, in particular, for potential field $a(x) = d\varphi(x)$ this will be a differential the equation

$$\Delta\varphi(x) = 0$$

Indeed, the value of $\Delta\varphi(x) = d \star d\varphi(x)$ characterizes the difference flow pressures on extremely small elements of level surfaces, obtained by gradient transfer in an extremely small element volume, namely,

$$\begin{aligned} \int_{\Delta V} \langle d \star d\varphi(x), dx^n \rangle = \\ \int_{\Delta\Phi_2} \langle \star d\varphi(x), dx^{n-1} \rangle - \int_{\Delta\Phi_1} \langle \star d\varphi(x), dx^{n-1} \rangle \end{aligned}$$

where the absence of pressure on the side surface is taken into account an extremely small volume element ΔV formed by note the extremely small area $\Delta\Phi_1$ of the surface level $\varphi(x) = c$ per pad $\Delta\Phi_2$ surface level $\varphi(x) = c + h$ when it is transferred by a gradient vector field. Therefore the harmony of the level function provides the absence of a pressure difference, i.e. the balance (stability) the corresponding stream. In the general case, when $a(x) = k(x)d\varphi(x)$, similar reasoning allows in as a criterion for the equilibrium of the flow, get the differential equation $d \star k(x)d\varphi(x) = k(x)\Delta\varphi(x) + (\nabla k(x), \nabla\varphi(x)) = 0$.

Let's now consider a variational problem, the solution of which will allow we need to get locally stable (minimal) flows. As we will take the value of the

curvilinear flow stability functional integral $\int \langle a(x), dx \rangle$ defined on a finite line segment of the General position. Since we are interested in the local criterion for the stability of the flow, then you should take the maximum small segment Δx and solve the variational equation

$$\text{Loc. var } \int \langle a(x), dx \rangle = \delta \int_x^{x+\Delta x} \langle a(x), dx \rangle = \delta \langle a(x + \Delta x) - a(x), \Delta x \rangle = 0$$

Thus, the integral variational problem it is reduced to a variational problem for minimum convolution increments. First, let the vector Δx be collinear with the vector $a^*(x)$. Then the equation $\delta \langle a(x + \Delta x) - a(x), \Delta x \rangle = \pm(|a(x + \Delta x)| - |a(x)|)|\Delta x| = 0$ is performed, if $|a(x + \Delta x)| = |a(x)|$, it follows that: the condition for performing the variational equation is constancy module of the covector field $a(x)$. Then let the vector Δx orthogonal to the vector $a^*(x)$. Then the equation $\delta \langle a(x + \Delta x) - a(x), \Delta x \rangle = \delta \langle a(x + \Delta x), \Delta x \rangle = 0$ is executed if the covector $a(x + \Delta x)$ is parallel to the $a(x)$ covector, which means that: a condition of the variational equations is holonomy of the field $a(x)$. Now let the vector Δx have a general position and the covector field $a(x)$ is singular and holonomic, i.e. $a(x) = k(x)d\varphi(x)$, where $k(x) = |\nabla\varphi(x)|^{-1}$. Then, since $a(x + \Delta x) = a(x) + \sum_n \langle \nabla a_j(x), \Delta x \rangle dx^j = a(x) + \nabla_{\Delta x} a(x)$, then the convolution increment is equal to the value of the quadratic form $\langle a(x + \Delta x) - a(x), \Delta x \rangle = \langle \nabla_{\Delta x} a(x), \Delta x \rangle = (H\Delta x, \Delta x)$, where the operator H in matrix representation is defined by non-diagonal elements $k(x) \frac{\partial^2 \varphi(x)}{\partial x_i \partial x_j} + \left(\frac{\partial k(x)}{\partial x_i} \frac{\partial \varphi(x)}{\partial x_j} + \frac{\partial k(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_i} \right)$ and diagonal elements $k(x) \frac{\partial^2 \varphi(x)}{\partial x_i^2} + \left(\frac{\partial k(x)}{\partial x_i} \frac{\partial \varphi(x)}{\partial x_i} \right)$. Thus, the variational equation $\delta(H\Delta x, \Delta x) = 0$ reduces our the task of finding the minimum self-adjoint operator. In this we remind [10] that every self-adjoint operator C up to reflections can be decomposed into the A C operator zero trace and positive definite operator B , i.e. $C = \pm(A + B)$. Indeed, it is enough to give the operator matrix C to the diagonal view and decompose its diagonal elements. In at the same time, for quadratic forms, the inequality is satisfied $(Ax, x) < ((A + B)x, x)$, hence the self-adjoint operator with the zero trace will be the minimum operator and the solution $\text{tr } H = k(x)\Delta\varphi(x) + (\nabla k(x), \nabla\varphi(x)) = 0$, where $k(x) = |\nabla\varphi(x)|^{-1}$, is necessary and sufficient condition of local extremality of the stability functional flow's. Note also that the same solution has a variational the problem for the extremum of the integral $\int \langle \star a(x), dx^{n-1} \rangle$ defined on the end site arbitrary $(n-1)$ -

plane in General position. Really, variational equation

$$\text{Loc. var } \int \langle \star a(x), dx^{n-1} \rangle = \delta \langle \star a(x + \Delta x) - \star a(x), \star \Delta x \rangle = 0$$

does not change statements of the variational problem, since the convolution of a vector with the covector is equal to the convolution of their dualizations.

Let now in the Euclidean space of the positive signature, where we do not distinguish vectors from covectors, a field of the 1-form $d\varphi(x)$ is given. Let's consider a single derivative of it holonomic field $n(x) = k(x)d\varphi(x)$, where $k(x) = 1/|d\varphi(x)|$, and calculate the area S of the surface Φ^{n-1} , which is a bounded part of the surface level of the $\varphi(x)$ function. We show that

$$S(\Phi^{n-1}) = \int_{\Phi^{n-1}} (\star n(x), dx^{n-1}).$$

Indeed, the unit vector $n(x)$ is orthogonal to the plane, tangent to the level surface of the $\varphi(x)$ function at the point x a single polyvector $\star n(x)$ is collinear to an extremely small one polyvector $\Delta \pi^{n-1}(x)$, and so the area the latter is equal to their scalar product $(\star n(x), \Delta \pi^{n-1}(x))$, which means that

$$S(\Phi^{n-1}) = \sum_{\Phi^{n-1}} S(\Delta \pi^{n-1}) = \int_{\Phi^{n-1}} (\star n(x), dx^{n-1}).$$

Observe also, that

$$\int_{\Delta V} \langle d \star n(x), dx^n \rangle = S(\Delta \Phi_2) - S(\Delta \Phi_1),$$

where is the platform $\Delta \Phi_2$ is formed by an orthogonal transfer of the site $\Delta \Phi_1$ for the smallest possible distance from the surface level Φ_1 to the surface of the level Φ_2 , and therefore

$$\lim_{\Delta V \rightarrow 0} \frac{S(\Delta \Phi_2) - S(\Delta \Phi_1)}{\Delta V} = d \star n(x).$$

However, it is easy to show that the local solution of the variational equations

$$\text{Loc. var } \int_{\Phi^{n-1}} (\star n(x), dx^{n-1}) = 0$$

leads to the same differential equation of minimality $d \star n(x) = 0$. Thus, locally minimal parameterized surfaces of Euclidean space with zero the average curvature and surface level of functions satisfying the minimality equation is equivalent

to. However, this equivalence also follows from the well-known fact that the average curvature of the level surface specified in n -dimensional Euclidean space, calculated using the formula

$$H(x) = -\frac{1}{n} d \star \left(\frac{d\varphi(x)}{|\nabla\varphi(x)|} \right)$$

In addition, as a result of the study, we obtained the duality of such concepts such as minimal surfaces and minimal (maximum steady) flow.

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**THE EXACT SOLUTIONS TO THE GRAVITATIONAL
CONTRACTION IN COMOVING COORDINATE SYSTEM**

Ying-Qiu Gu

School of Mathematical Science, Fudan University
Shanghai 200433, China
yqgu@fudan.edu.cn

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Abstract

The gravitational collapse of a star is a warmly discussed but still puzzling problem, which not only involves the dynamics of fluid, but also the subtle coordinate transformation. In this paper, we give a detailed investigation on this problem, and reach the results: (I). The comoving coordinate system for the stellar system is only compatible with the zero-pressure free falling particles. (II). For the free falling dust, there are three kind of solutions respectively corresponding to the oscillating, the critical and the open trajectories. The solution of Oppenheimer and Snyder is the critical case. (III). All solutions are exactly derived. There is a new kind singularity in the solution, but its origin is unclear.

Keywords: *gravitational collapse, comoving coordinate system, exact solution, singularity*

The gravitational collapse of a star is a warmly discussed but still puzzling problem, which not only involves the dynamics of fluid, but also the subtle coordinate transformation. Based on the "Comoving Coordinate System"[1, 2], J. Oppenheimer and H. Snyder discussed an enlighten model for free falling dust[3]. In the paper [3] they said: "To investigate this question we will solve the field equations with the limiting form of the energy-momentum tensor in which the pressure is zero. When the pressure vanishes there are no static solutions to the field equations except when all components of T^μ_ν vanishes". "we are supposing that the relationships between the T^μ_ν do not admit any stationary solutions, and therefore exclude this possibility." So they actually assumed the singularity exists, and ignored the dominant gravitational potential itself is a more powerful source of pressure than the electromagnetic interaction[4, 5].

In this paper, we give a more detailed investigation on this problem. From the calculation, we find that:

1. The comoving coordinate system for the stellar system is only compatible with the zero-pressure free falling particles, so it is not a globally valid coordinate system for the case.
2. Similarly to the cosmological model, for the free falling dust we also have three kind of solutions, namely, the oscillating, the critical and the open trajectories. The solution of Oppenheimer and Snyder corresponds to the critical one.
3. All exact solutions to the free falling dust can be manifestly derived, but the origin of a new kind singularity is not clarified.

The line element in the comoving coordinate system is given by[1, 2]

$$ds^2 = dt^2 - e^\lambda dr^2 - e^w (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (1)$$

In this case, we have the 4-dimensional speed of the fluid $U^\mu = (1, 0, 0, 0)$, and then the energy-momentum tensor becomes

$$T_{\mu\nu} = \text{diag}(\rho, -Pe^\lambda, -Pe^w, -Pe^w \sin^2 \theta). \quad (2)$$

In the comoving coordinate system, we find all exact solutions to Einstein's field equation can be solved by means of variable substitution. That is, we firstly express all variables as functions of w and its derivatives, and then solve a simplified equation for w .

By the continuity equation $U_\mu T^{\mu\nu}_{;\nu} = 0$, manifestly

$$U^\mu \partial_\mu \rho + (\rho + P)U^\mu_{;\mu} = 0,$$

we obtain

$$\frac{1}{2} \partial_t \lambda + \partial_t w + \frac{\partial_t \rho}{\rho + P} = 0. \quad (3)$$

For the barotropic fluid, defining

$$B(\rho) = \int \frac{d\rho}{\rho + P(\rho)}, \quad (4)$$

then we can solve for λ from (3) as

$$\lambda = -2(w + B) + \alpha(r), \quad (5)$$

where α is a function to be determined. Denoting $\kappa = 8\pi G$, substituting (5) into the Einstein's field equation $G_{tr} + \kappa T_{tr} = 0$, namely

$$\partial_{tr}^2 w + \frac{1}{2}(\partial_t w \partial_r w - \partial_t \lambda \partial_r w) = 0, \quad (6)$$

we obtain

$$B = -\frac{3}{2}w - \ln(\partial_r w) + \beta(r), \quad (7)$$

where $\beta(r)$ is also a function to be determined. Substituting (7) into (5), we obtain λ expressed by w

$$\lambda = w + 2\ln(\partial_r w) - \ln[4(1 + \gamma)], \quad (8)$$

in which $\gamma(r) = \frac{1}{4}e^{2\beta-\alpha} - 1$ is introduced for convenience. The effect of γ is similar to that of the dimensionless energy density Ω in cosmology as shown below, which determines the different evolving fate of the star.

Substituting the above results into $G_{tt} + \kappa T_{tt} = 0$, we obtain ρ expressed by w

$$\rho = -\frac{1}{\kappa\partial_r w} \left[(\gamma\partial_r w + 2\partial_r \gamma)e^{-w} + \partial_t w \left(\frac{3}{4}\partial_t w \partial_r w + \partial_{tr}^2 w \right) \right]. \quad (9)$$

Taking $\partial_t \rho$ as variable independent of ρ and substituting (9) into $G_{rr} + \kappa T_{rr} = 0$, we obtain $\partial_t \rho$ expressed by w

$$\partial_t \rho = \frac{1}{2\kappa(\partial_r w)^2} (2\partial_{tr}^2 w + 3\partial_t w \partial_r w) (\partial_t^2 w \partial_r w - \partial_t w \partial_{tr}^2 w + 2\partial_r \gamma e^{-w}). \quad (10)$$

Substituting (9) into (10), we obtain the equation of consistence

$$\partial_{t^2 r}^3 w = -\frac{3}{2}\partial_t w \partial_{tr} w + \partial_r (\gamma e^{-w}). \quad (11)$$

Integrating (11) with respect to r , we obtain

$$\partial_t^2 w = -\frac{3}{4}(\partial_t w)^2 + \gamma e^{-w} + \delta(t), \quad (12)$$

where $\delta(t)$ is a function to be determined. In [3] Oppenheimer and Snyder assumed $\gamma = \delta = 0$. As shown below, their solution only corresponds to the critical case.

Substituting (11) and (12) into $G_{\theta\theta} + \kappa T_{\theta\theta} = 0$, we find the physical meaning of δ as

$$\delta(t) = -\kappa P. \quad (13)$$

Relation (13) has an important implication, that is, the metric (1) is unsuitable to describe the realistic process of the star contraction, because in this case we definitely have $\partial_r P \neq 0$, which contradicts (13). The comoving coordinate system is usually a local coordinate system in heavily curved space-time, and only in cosmology (1) is globally valid, because in this case $P = P(t)$. This shows the limitation of the comoving metric (1), we will give more explanation in the next.

By (13), we find the only consistent matter in the space-time (1) is the fluid with zero pressure. Of course, this model is extremely idealized, because the particles passing across the center will continue to move outward, and the collision among particles leads to high pressure and temperature. Such high pressure will balance the contraction[4, 5].

Now we solve (12) with $\delta = -\kappa P = 0$. Setting $w = \frac{4}{3}\ln(u)$, by (12) we obtain

$$\partial_t^2 u = \frac{3\gamma}{4}u^{-\frac{1}{3}}, \quad (\partial_t u)^2 = \frac{9}{4} \left(\mathcal{E} + \gamma u^{\frac{2}{3}} \right), \quad (14)$$

where $\mathcal{E}(r)$ is a function to be determined. Let $u = v^{\frac{3}{2}}$, that is, making transformation

$$w = \ln(v^2), \quad v = e^{\frac{w}{2}}, \quad (15)$$

we obtain

$$(\partial_t v)^2 = \frac{\mathcal{E} + \gamma v}{v}. \quad (16)$$

Substituting (16) into (9), we learn the physical meaning of $\mathcal{E}$

$$\partial_r \mathcal{E} = \kappa \rho v^2 \partial_r v, \quad \mathcal{E}(r) = \mathcal{E}_0 + \int_0^r \kappa \rho v^2 \partial_r v dr, \quad (17)$$

where $\mathcal{E}_0$ is a constant. $\mathcal{E}(r)$ corresponds to the comoving mass in the ball of radius r . Therefore, the quantity $\int_0^r \rho v^2 \partial_r v dr$ is conserved. By their

physical meanings, we certainly have $\partial_r v > 0$ and $\partial_r \mathcal{E} \geq 0$. From (17), we find v can be regarded as new spatial coordinate to replace r . Substituting $\partial_r v$ into (8), we obtain

$$e^{\frac{\lambda}{2}} = \frac{\partial_r \mathcal{E}}{\kappa \rho v^2 \sqrt{1 + \gamma}}, \quad \rho = \frac{\partial_r \mathcal{E} e^{-\frac{\lambda}{2}}}{\kappa v^2 \sqrt{1 + \gamma}}. \quad (18)$$

Among the parameters $(\alpha, \beta, \gamma, \delta, \mathcal{E})$, only $(\gamma, \mathcal{E})$ are independent functions. Their relations are given by

$$\beta = \ln \frac{2\partial_r \mathcal{E}}{\kappa}, \quad B = \ln \rho, \quad \alpha = 2 \ln \frac{\partial_r \mathcal{E}}{\kappa \sqrt{1 + \gamma}}, \quad \delta = \kappa P, \quad (19)$$

where β is derived by substituting (16) into (7), and α is derived by the definition of γ .

If $\gamma = 0$, we obtain the solution derived in [3] as follows

$$v = \frac{1}{2} \sqrt[3]{18\mathcal{E}|t - t_0(r)|^2}. \quad (20)$$

The curvature at $t = t_0(r)$ is singular. Substituting (20) into (8), we obtain

$$e^{\frac{\lambda}{2}} = \frac{|\partial_r \mathcal{E}(t - t_0) - 2\mathcal{E}\partial_r t_0|}{\sqrt[3]{12\mathcal{E}^2|t - t_0|}}. \quad (21)$$

Substituting (20) and (21) into (18), we find

$$\rho = \frac{4\partial_r \mathcal{E} e^{-\frac{\lambda}{2}}}{3\kappa \sqrt[3]{12\mathcal{E}^2(t - t_0)^4}} = \frac{4\partial_r \mathcal{E}}{3\kappa |[\partial_r \mathcal{E}(t - t_0) - 2\mathcal{E}\partial_r t_0](t - t_0)|}. \quad (22)$$

At the surface $t = t_0(r)$ we have a singularity, which corresponds to the matter on this surface is passing through the stellar center. This space-time is bouncing. The phase trajectories of (16) are displayed in Figure 1. As shown below, if $\gamma < 0$, the space-time is oscillating, and if $\gamma \geq 0$, the space-time is bouncing.

In the case $\gamma < 0$, letting $\gamma = -\mathcal{E}\eta^{-2}$, then we get the solution

$$\frac{1}{2}\eta^2 \arccos \frac{\eta^2 - 2v}{\eta^2} - \sqrt{v(\eta^2 - v)} = \frac{\sqrt{\mathcal{E}}}{\eta} |t - t_0|. \quad (23)$$

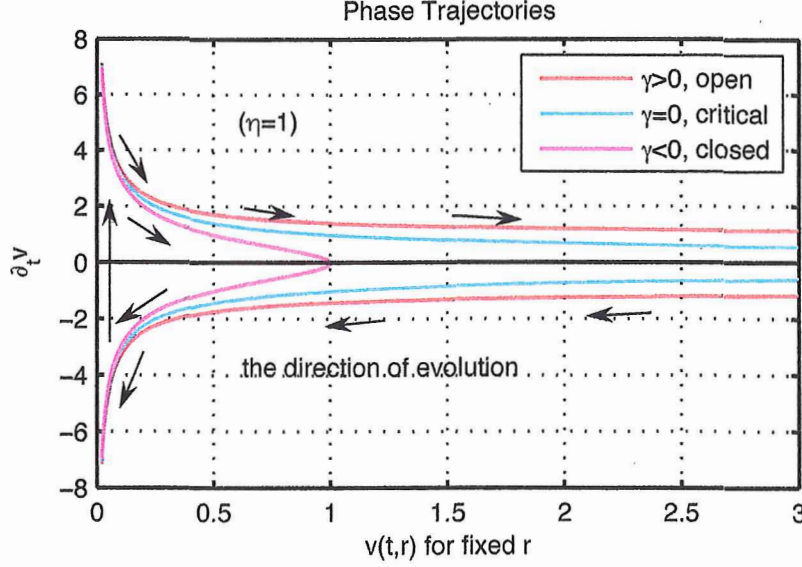


Figure 1: There are three kind of trajectories for the free falling dust. If $\gamma < 0$, which means the kinetic energy of the particles is small, the orbits is oscillating. If $\gamma \geq 0$, the orbits is open

The space-time is oscillating. Solving for $\partial_r v$ from (23) and substituting it into (8), we obtain

$$e^{\frac{\lambda}{2}} = \frac{\sqrt{v(\eta^2 - v)}\Delta}{2\eta v\sqrt{\mathcal{E}(\eta^2 - \mathcal{E})}}, \quad (24)$$

in which

$$\Delta = [(\eta\partial_r\mathcal{E} - 6\mathcal{E}\partial_r\eta)|t - t_0| - 2\text{sign}(t - t_0)\mathcal{E}\eta\partial_r t_0] + 4\sqrt{\mathcal{E}v^2\eta\partial_r\eta}. \quad (25)$$

In the case $\gamma > 0$, letting $\gamma = \mathcal{E}\eta^{-2}$, we obtain the solution

$$\eta^2 \ln \frac{\eta}{\sqrt{v} + \sqrt{\eta^2 + v}} + \sqrt{v(\eta^2 + v)} = \frac{\sqrt{\mathcal{E}}}{\eta} |t - t_0|. \quad (26)$$

The space-time is bouncing. Solving $\partial_r v$ from (26) and substituting it into

(8), we obtain

$$e^{\frac{\lambda}{2}} = \frac{\sqrt{v(\eta^2 + v)}\Delta}{2\eta v\sqrt{\mathcal{E}(\eta^2 + \mathcal{E})}}. \quad (27)$$

When $t \rightarrow t_0$, the variables have the following asymptotic behavior in all cases,

$$v = e^{\frac{w}{2}} \rightarrow \frac{1}{2} \sqrt[3]{18\mathcal{E}|t - t_0(r)|^2}. \quad (28)$$

$$e^{\frac{\lambda}{2}} \rightarrow \frac{|\partial_r \mathcal{E}(t - t_0) - 2\mathcal{E}\partial_r t_0|}{\sqrt[3]{12\mathcal{E}^2|t - t_0|}\sqrt{1 + \gamma}}. \quad (29)$$

$$\rho \rightarrow \frac{4\partial_r \mathcal{E}\sqrt{1 + \gamma}}{3\kappa|[\partial_r \mathcal{E}(t - t_0) - 2\mathcal{E}\partial_r t_0](t - t_0)|}. \quad (30)$$

From the above calculation, we find the singularity at $t = t_0$ implies the free falling particles pass across the stellar center, so it is a singularity caused by the simplification of the matter model, but not a real one. The dynamics for the pressure-free fluid is ill-posed in mathematics, which results in the zero speed of sound and ' δ -waves'. The solutions just provide some intuitions.

There are another kind singularity hides in the solutions. In the critical case $\gamma = 0$, (21) and (22) become singular at time

$$t = t_0 - \frac{2\mathcal{E}\partial_r t_0}{\partial_r \mathcal{E}}. \quad (31)$$

If $t_0(r)$ is not a constant, then $t \neq t_0$, the singularity is different from the one mentioned above. $t_0(r) = \text{const.}$ means a moving mass shell. So whether this singularity is caused by the globally invalidity of the comoving coordinate system or by the ill-posed model is not clear. Strangely enough, the function w or v is normal at this time.

In the comoving coordinate system, the spatial coordinates are defined by the trajectories of a set of particles, and the temporal coordinate

is defined by the proper time of the particles[2]. In cosmology, since the space is isotropic and homogeneous, this coordinate system works well. However, in the heavily curved space-time, the comoving coordinate system is actually the Einstein's lift moving along a geodesic, which is only valid locally and requires the lift to be an infinitesimal volume. The Gaussian normal coordinate system or the semi-geodesic coordinate stem is also valid locally, because the global simultaneous hypersurface is valid only in the neighborhood of the initial Cauchy hypersurface. The isodistant translation of the hypersurface will deform soon, so that the metric $ds^2 = dt^2 + g_{kl}dx^k dx^l$ becomes invalid. The generally and globally valid coordinate system should be Gu's natural coordinate system(NCS)[6]

$$ds^2 = g_{tt}dt^2 - \bar{g}_{kl}dx^k dx^l, \quad d\tau = \sqrt{g_{tt}}dt = f_t^0 dt, \quad dV = \sqrt{\bar{g}}d^3x. \quad (32)$$

In which ds is the 4- d length of line element, $d\tau$ is the Newton's absolute cosmic time element and dV is the absolute volume element of space at time t . g_{tt} represents the gravity, which cannot be merged into time coordinate t generally. The NCS generally exists and the global simultaneity is unique. Only in NCS can we clearly establish the Hamiltonian formalism of particles[7].

In summary, from the above calculation and analysis, we find that the gravitational contraction is not so simple as it looks like. The comoving coordinate system for the stellar system is only compatible with the zero-pressure free-falling particles, so it is not globally valid coordinate system. The free falling dust is just an idealized model, the final truth relies on the detailed analysis of dynamics with realistic equation of state of matter[4, 5].

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**MINIMAL FREE RESOLUTION OF A FAMILY OF
GENERIC MONOMIAL IDEALS**

J. William Hoffman

Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803
hoffman@math.lsu.edu
and

Haohao Wang

Department of Mathematics
Southeast Missouri State University
Cape Girardeau, MO 63701
hwang@semo.edu

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Abstract

In this paper, we study a family of generic monomial ideals generated by the monomial parametrization of rational surfaces. First, we describe the necessary and sufficient conditions for this family of monomial ideals to be generic. Then, we classify all the possible minimal graded free resolutions of this family.

Keywords: Monomial ideal; Minimal Free Resolution.

1 Introduction

In this paper, we investigate the family of monomial ideals given by the bihomogeneous forms of bidegree (m, n) in the standard bigraded ring $S = \mathbb{K}[s_0, s_1; t_0, t_1]$ with $\deg(s_i) = (1, 0)$, $\deg(t_i) = (0, 1)$ over an infinite field $\mathbb{K}$:

$$I = \langle s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0}, s_0^{a_1} s_1^{m-a_1} t_0^{b_1} t_1^{n-b_1}, s_0^{a_2} s_1^{m-a_2} t_0^{b_2} t_1^{n-b_2}, s_0^{a_3} s_1^{m-a_3} t_0^{b_3} t_1^{n-b_3} \rangle, \quad (1)$$

where the ideal I is generated by the monomial parametrization of rational surfaces $\varphi : \mathbb{P}_{\mathbb{K}}^1 \times \mathbb{P}_{\mathbb{K}}^1 \rightarrow \mathbb{P}_{\mathbb{K}}^3$ of the following form: $\varphi = \varphi(s_0, s_1; t_0, t_1)$ where

$$\varphi = (s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0}, s_0^{a_1} s_1^{m-a_1} t_0^{b_1} t_1^{n-b_1}, s_0^{a_2} s_1^{m-a_2} t_0^{b_2} t_1^{n-b_2}, s_0^{a_3} s_1^{m-a_3} t_0^{b_3} t_1^{n-b_3}), \quad (2)$$

and

$$\gcd(s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0}, s_0^{a_1} s_1^{m-a_1} t_0^{b_1} t_1^{n-b_1}, s_0^{a_2} s_1^{m-a_2} t_0^{b_2} t_1^{n-b_2}, s_0^{a_3} s_1^{m-a_3} t_0^{b_3} t_1^{n-b_3}) = 1.$$

The motivation for examining this family of monomial parametrizations of rational surfaces and their corresponding monomial ideals originates in the implicitization problem and its connection to the Rees algebra. Many works have been devoted to this subject [4], [5], [6], [7], [8], [9], and [15]. We are particularly interested in the minimal free resolutions of this family of monomial ideals. There is a large literature on ideals generated by monomials and the construction of their minimal free resolutions [1], [2], [3], [14], [18]. Many papers studied the Taylor resolution of Eisenbud [11], Lyubeznik's subcomplex [12], and the Scarf complex [17] and its variants. However, in general, the Taylor resolution does not give a minimal free resolution, and the Scarf complex and its variants are not free. To our knowledge, it remains an open question to classify all the possible minimal graded free resolutions of the family of generic monomial ideals generated by the monomial parametrization of rational surfaces.

This paper is a sequel to the paper [16], where we studied a family of a special monomial parametrizations of rational surfaces, and we derived a few structural properties related to the corresponding monomial ideals

generated by these parametrizations. However, the ideals studied in [16] do not have the genericity property, that is, that no variable appears with the same non-zero exponent in two distinct minimal generators of I . It is natural to ask: “what are the necessary and sufficient conditions for the monomial ideal I to be generic?” and “what are the structures of these generic monomial ideals?”. The goal of this paper is to provide explicit answers to these two questions in a specific family. First, we describe the necessary and sufficient conditions for the family of monomial ideals I to be generic. Then, we classify all the possible minimal graded free resolutions of this family of generic monomial ideals.

This paper is structured as the following. We begin in Section 2 with the definition of generic monomial ideals. We formulate in Theorem 2.2 the necessary and sufficient conditions for the ideal I to be generic, and we conclude that a generic monomial ideal must be in one of the three forms described in Corollary 2.4. Then, in Section 3, we classify all the possible minimal graded free resolutions of this family of generic monomial ideals. Our approach is to create resolutions of generic monomial ideals from the exponents of the defining monomials in one of the three forms described in Corollary 2.4. Examples are provided to illustrate our results.

2 Generic Monomial Ideals

We start this section by first recalling the definition of a monomial support and a generic monomial ideal.

Let M be a monomial ideal minimally generated by monomials $m_1, \dots, m_r$ in a polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ over a field $\mathbb{K}$. For a subset $\sigma \subseteq \{1, \dots, r\}$, we set $m_\sigma := \text{lcm}(m_i | i \in \sigma)$, and let $\mathbf{a}_\sigma := \deg m_\sigma \in \mathbb{N}^n$ denote the exponent vector of m_σ . Here $m_\emptyset = 1$. For a monomial $x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$, we set $\deg(x_i) = a_i$ and $\text{supp}(x^{\mathbf{a}}) := \{i | a_i \neq 0\} \subseteq \{1, \dots, n\}$ the support of $x^{\mathbf{a}}$. We say a monomial $m \in S$ **strictly divides** $m' \in \mathbb{K}[x_1, \dots, x_n]$, if m divides m' and $\text{supp}(m'/m) = \text{supp}(m')$.

Definition 2.1. A monomial ideal $M = \langle m_1, \dots, m_r \rangle$ is called **generic** if whenever two distinct minimal generators m_i and m_j have the same positive

degree in some variable x_s , then there is a third generator m_i which strictly divides $m_{i,j} = \text{lcm}(m_i, m_j)$. A monomial ideal $M = \langle m_1, \dots, m_r \rangle$ is called **strongly generic** if no two distinct minimal generators m_i and m_j have the same positive degree in some variable x_s .

In Theorem 2.2 below, we provide the necessary and sufficient conditions for the ideals I in (1) generated by the rational parametrization (2) to be generic.

Theorem 2.2. *An ideal I in (1) generated by the parametrization (2) for rational surfaces is generic if and only if it is strongly generic. This is so if and only if a_0, a_1, a_2, a_3 are distinct elements of $\{0, 1, \dots, m\}$ and b_0, b_1, b_2, b_3 are distinct elements of $\{0, 1, \dots, n\}$ and moreover, without loss of generality assuming $a_0 > a_1$ and $a_2 > a_3$, the dot product*

$$\begin{aligned} & (a_1, m - a_0, \min(b_0, b_1), \min(n - b_0, n - b_1)) \cdot \\ & (a_3, m - a_2, \min(b_2, b_3), \min(n - b_2, n - b_3)) = 0. \end{aligned}$$

Note that to be a rational parametrization, we are assuming that the GCD of the four monomials is 1.

Proof. ($\Rightarrow$) Suppose the ideal I in (1) generated by the rational surface parametrization (2) is generic. Since every monomial ideal has a unique minimal finite set of monomial generators [13, Lemma 1.2], we will show that the exponents of the generators satisfy the given conditions, and I is strongly generic.

Let $s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0}$, $s_0^{a_1} s_1^{m-a_1} t_0^{b_1} t_1^{n-b_1}$ are two distinct monomial generators of I , that is $(a_0, b_0) \neq (a_1, b_1)$. We claim that if I is generic then $a_0 \neq a_1$ and $b_0 \neq b_1$. To prove this claim, we will show that if I is generic then it is impossible for either $a_0 = a_1$ or $b_0 = b_1$. In what follows, for the sake of simplicity, we will only show that it is impossible for $a_0 = a_1$; the symmetric case can be treated exactly the same way.

Suppose $a_0 = a_1$. Without loss of generality, assume $b_0 < b_1$ (the argument below applies if $b_1 < b_0$), then

$$\text{lcm}(s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0}, s_0^{a_1} s_1^{m-a_1} t_0^{b_1} t_1^{n-b_1}) = s_0^{a_0} s_1^{m-a_0} t_0^{b_1} t_1^{n-b_0}.$$

By Definition 2.1 of generic, there is a third monomial generator $s_0^\alpha s_1^{m-\alpha} t_0^\beta t_1^{n-\beta}$ such that

$$s_0^\alpha s_1^{m-\alpha} t_0^\beta t_1^{n-\beta} \text{ strictly divides } s_0^{a_0} s_1^{m-a_0} t_0^{b_1} t_1^{n-b_0}.$$

If $0 < a_0 < m$, then the definition of strictly divides yields that $a_0 - \alpha > 0$ and $(m - a_0) - (m - \alpha) = \alpha - a_0 > 0$, which is absurd. If $a_0 = 0$, then $m - a_0 = m > 0$. The division forces $\alpha = 0$ and the exponent of s_1 disappears after division, so the division is not strict. Similar argument for $a_0 = m$. This shows that if I is generic, then a_0, a_1, a_2, a_3 are distinct elements of $\{0, 1, \dots, m\}$ and b_0, b_1, b_2, b_3 are distinct elements of $\{0, 1, \dots, n\}$, hence it is strongly generic in the sense of Definition 2.1.

Without loss of generality, assume that $a_0 > a_1$ and $a_2 > a_3$. To see the dot product condition note that the GCDs of the first two generators and the last two generators are

$$\begin{aligned} G_1 &= \gcd(s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0}, s_0^{a_1} s_1^{m-a_1} t_0^{b_1} t_1^{n-b_1}) \\ &= s_0^{a_1} s_1^{m-a_0} t_0^{\min(b_0, b_1)} t_1^{\min(n-b_0, n-b_1)}, \\ G_2 &= \gcd(s_0^{a_2} s_1^{m-a_2} t_0^{b_2} t_1^{n-b_2}, s_0^{a_3} s_1^{m-a_3} t_0^{b_3} t_1^{n-b_3}) \\ &= s_0^{a_3} s_1^{m-a_2} t_0^{\min(b_2, b_3)} t_1^{\min(n-b_2, n-b_3)}. \end{aligned}$$

To be the parametrization of a rational surface, $\gcd(G_1, G_2) = 1$, which means that if the exponent of a variable of G_1 is non-zero, then the exponent of the same variable of G_2 must be zero, that is, the dot product

$$\begin{aligned} &(a_1, m - a_0, \min(b_0, b_1), \min(n - b_0, n - b_1)) \cdot \\ &(a_3, m - a_2, \min(b_2, b_3), \min(n - b_2, n - b_3)) = 0. \end{aligned}$$

($\Leftrightarrow$) If the minimal generators of monomial ideal I is in the form:

$$\begin{aligned} I &= \langle s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0}, s_0^{a_1} s_1^{m-a_1} t_0^{b_1} t_1^{n-b_1}, s_0^{a_2} s_1^{m-a_2} t_0^{b_2} t_1^{n-b_2}, s_0^{a_3} s_1^{m-a_3} t_0^{b_3} t_1^{n-b_3} \rangle \\ &\text{where } a_i \in \{0, 1, \dots, m\}, b_i \in \{0, 1, \dots, n\}, 0 \leq i \leq 3, \\ &\text{with } a_0 > a_1 \neq a_2 > a_3, b_0 \neq b_1 \neq b_2 \neq b_3, \end{aligned}$$

then I is strongly generic in the sense of Definition 2.1. We only need to show that the GCD of the generators in I is 1, and hence I is generated by the rational surface parametrization (2).

Let G_1, G_2 be the GCDs of the first two generators and the last two generators. The conditions $a_0 > a_1 \neq a_2 > a_3$ imply that $(a_1, m - a_0, \min(b_0, b_1), \min(n - b_0, n - b_1))$ and $(a_3, m - a_2, \min(b_2, b_3), \min(n - b_2, n - b_3))$ are the exponents of G_1 and G_2 . The dot product

$$\begin{aligned} & (a_1, m - a_0, \min(b_0, b_1), \min(n - b_0, n - b_1)) \cdot \\ & (a_3, m - a_2, \min(b_2, b_3), \min(n - b_2, n - b_3)) = 0 \end{aligned}$$

yields that if the exponent of a variable of G_1 is non-zero, then the exponent of the same variable of G_2 must be zero, and hence $\gcd(G_1, G_2) = 1$. Thus,

$$\gcd(s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0}, s_0^{a_1} s_1^{m-a_1} t_0^{b_1} t_1^{n-b_1}, s_0^{a_2} s_1^{m-a_2} t_0^{b_2} t_1^{n-b_2}, s_0^{a_3} s_1^{m-a_3} t_0^{b_3} t_1^{n-b_3}) = 1.$$

Therefore, I is a strongly generic, hence generic, monomial ideal generated by the rational surface parametrization given in (2). $\square$

Example 2.3. Applying Theorem 2.2, we construct a generic monomial ideal corresponding to a monomial parametrization for a rational surface, where $I = \langle s_0^2 s_1^4 t_0^2 t_1^5, s_0 s_1^5 t_0 t_1^6, s_0^6 t_0^7, s_1^6 t_1^7 \rangle$.

Note $(m, n) = (6, 7)$, $(a_0, a_1, a_2, a_3) = (2, 1, 6, 0)$ and $(b_0, b_1, b_2, b_3) = (2, 1, 7, 0)$ satisfy the conditions $a_0 > a_1 \neq a_2 > a_3$, $b_0 \neq b_1 \neq b_2 \neq b_3$, and

$$\begin{aligned} & (a_1, m - a_0, \min(b_0, b_1), \min(n - b_0, n - b_1)) \cdot \\ & (a_3, m - a_2, \min(b_2, b_3), \min(n - b_2, n - b_3)) \\ & = (1, 4, 1, 5) \cdot (0, 0, 0, 0) = 0. \end{aligned}$$

Therefore, the ideal I is a strongly generic monomial ideal.

As a direct consequence of Theorem 2.2, we conclude that a generic monomial ideal must be in one of the three forms described in Corollary 2.4 below.

Corollary 2.4. Up to a change of variables, the generic ideal I given in

Theorem 2.2 is in one of the following three forms:

- (1) $I = \langle s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0}, s_0^{a_1} s_1^{m-a_1} t_0^{b_1} t_1^{n-b_1}, s_0^m t_0^n, s_1^m t_1^n \rangle,$
 $0 < a_1 < a_0 < m, 0 < b_0 \neq b_1 < n,$
- (2) $I = \langle s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0}, s_1^m t_0^{b_1} t_1^{n-b_1}, s_0^m t_0^n, s_0^{a_3} s_1^{m-a_3} t_1^n \rangle,$
 $0 < a_0 \neq a_3 < m, 0 < b_0 \neq b_1 < n,$
- (3) $I = \langle s_0^{a_0} s_1^{m-a_0} t_1^n, s_1^m t_0^{b_1} t_1^{n-b_1}, s_0^m t_0^{b_2} t_1^{n-b_2}, s_0^{a_3} s_1^{m-a_3} t_0^n \rangle,$
 $0 < a_0 \neq a_3 < m, 0 < b_1 \neq b_2 < n.$

Proof. Recall in Theorem 2.2,

$$I = \langle s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0}, s_0^{a_1} s_1^{m-a_1} t_0^{b_1} t_1^{n-b_1}, s_0^{a_2} s_1^{m-a_2} t_0^{b_2} t_1^{n-b_2}, s_0^{a_3} s_1^{m-a_3} t_0^{b_3} t_1^{n-b_3} \rangle$$

where $a_i \in \{0, 1, \dots, m\}, b_i \in \{0, 1, \dots, n\}, 0 \leq i \leq 3,$
 with $a_0 > a_1 \neq a_2 > a_3, b_0 \neq b_1 \neq b_2 \neq b_3,$

Up to a change of variables, there are only three distinct cases.

Case 1: If all $a_1, m - a_0, \min(b_0, b_1), \min(n - b_0, n - b_1)$ are non-zero, then all $a_3, m - a_2, \min(b_2, b_3), \min(n - b_2, n - b_3)$ are zero, and

$$I = \langle s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0}, s_0^{a_1} s_1^{m-a_1} t_0^{b_1} t_1^{n-b_1}, s_0^m t_0^n, s_1^m t_1^n \rangle,$$

$$0 < a_1 < a_0 < m \text{ and } 0 < b_0 \neq b_1 < n.$$

Case 2: If three of $a_1, m - a_0, \min(b_0, b_1), \min(n - b_0, n - b_1)$ are non-zero, then the corresponding three of $a_3, m - a_2, \min(b_2, b_3), \min(n - b_2, n - b_3)$ are zero, and up to a change of variables, the ideal I is unique of the form

$$I = \langle s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0}, s_1^m t_0^{b_1} t_1^{n-b_1}, s_0^m t_0^n, s_0^{a_3} s_1^{m-a_3} t_1^n \rangle,$$

$$0 < a_0 \neq a_3 < m \text{ and } 0 < b_0 \neq b_1 < n.$$

Case 3: If two of $a_1, m - a_0, \min(b_0, b_1), \min(n - b_0, n - b_1)$ are non-zero, then the corresponding two of $a_3, m - a_2, \min(b_2, b_3), \min(n - b_2, n - b_3)$ are zero, and up to a change of variables, the ideal I is unique of the form

$$I = \langle s_0^{a_0} s_1^{m-a_0} t_1^n, s_1^m t_0^{b_1} t_1^{n-b_1}, s_0^m t_0^{b_2} t_1^{n-b_2}, s_0^{a_3} s_1^{m-a_3} t_0^n \rangle,$$

$$0 < a_0 \neq a_3 < m \text{ and } 0 < b_1 \neq b_2 < n.$$

□

3 Minimal Free Resolutions

In this section, we will classify the structure of minimal graded free resolutions of the generic monomial ideals I . Our approach is to create resolutions of generic monomial ideals from the exponents of the defining monomials in one of the three forms described in Corollary 2.4. First, we recall the definition of a Taylor's resolutions.

Definition 3.1. Let $I = \langle m_1, \dots, m_r \rangle$ be a monomial ideal in S with minimal generators. **Taylor's resolution**, is the free resolution which supports the full $(r - 1)$ -dimensional simplex Δ_I whose vertices are labeled by the monomials which generate I .

Remark 3.2. It is important to point out that Taylor's resolution may not be minimal. It is proved in [13, Lemma 6.4] that the Taylor complex is minimal if and only if for all faces $\sigma \in \Delta_I$ and all indices $i \in \sigma$, the monomials m_σ and $m_{\sigma \setminus i}$ are different.

We will use the following example to illustrate how to construct Taylor's resolution, and apply Remark 3.2 to determine if the Taylor's resolution is minimal.

Example 3.3. We continue with Example 2.3 where

$$I = \langle s_0^2 s_1^4 t_0^2 t_1^5, s_0 s_1^5 t_0^6 t_1^6, s_0^6 t_0^7, s_1^6 t_1^7 \rangle \subset S.$$

By Definition 3.1, the Taylor resolution for I is a three dimensional simplicial complex Δ_I . Consider the four minimal generators of I to be the vertices of Δ_I . Consider that all of the four points must be connected to each other by an edge, hence Δ_I has $\binom{4}{2} = 6$ edges, where the edge is identified as the LCM of the two vertices it connects. We continue with this procedure and see that Δ_I has four 2-dimensional faces where each face is identified as the LCM of the edges it contains; and one 3-dimensional face that is identified as the LCM of these four 2-dimensional faces.

The Taylor resolution is

$$0 \longrightarrow S_3 \xrightarrow{\varphi_3} S_2^4 \xrightarrow{\varphi_2} S_1^6 \xrightarrow{\varphi_1} S_0^4 \longrightarrow I \longrightarrow 0,$$

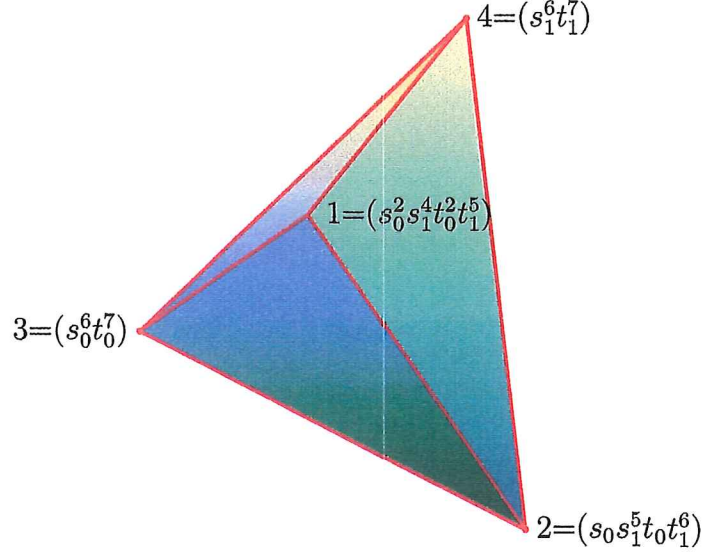


Figure 1: 3-dimensional simplex Δ_I whose vertices labeled by the monomial generators of $I = \langle s_0^2 s_1^4 t_0^2 t_1^5, s_0 s_1^5 t_0 t_1^6, s_0^6 t_0^7, s_1^6 t_1^7 \rangle$. Labelling the vertices 1, 2, 3, 4 in the order of the monomials, the face opposite to the vertex i gets orientation $(-1)^{i+1}$.

where the subindex of S_i denote the i -dimensional face, and

$$\begin{aligned}
 S_0 &= S(s_0^2 s_1^4 t_0^2 t_1^5, s_0 s_1^5 t_0 t_1^6, s_0^6 t_0^7, s_1^6 t_1^7) \\
 &\quad \text{denotes 4 vertices indexed 1, 2, 3, 4} \\
 S_1 &= S(s_0^2 s_1^5 t_0 t_1^6, s_0 s_1^4 t_0^2 t_1^5, s_0^2 s_1^6 t_0^2 t_1^7, s_0^6 s_1^5 t_0 t_1^6, s_0 s_1^6 t_0 t_1^7, s_0^6 s_1^6 t_0^7 t_1^7) \\
 &\quad \text{denotes 6 edges indexed [12], [13], [14], [23], [24], [34]} \\
 S_2 &= S(s_0^6 s_1^5 t_0 t_1^6, s_0^2 s_1^6 t_0^2 t_1^7, s_0^6 s_1^6 t_0^7 t_1^7, s_0^6 s_1^6 t_0^7 t_1^7) \\
 &\quad \text{denotes three 2-dimensional faces indexed [123], [124], [134], [234]} \\
 S_3 &= S(s_0^6 s_1^6 t_0^7 t_1^7) \\
 &\quad \text{denotes one 3-dimensional faces indexed [1234]}.
 \end{aligned}$$

Recall that elements of each module are labels of distinct faces, so if the same label appears multiple times in a module, then the resolution we are constructing is non-minimal. Immediately, we note that the monomials

correspond to the 2-dimensional faces $[134]$ and $[234]$ are the same, and hence this Taylor's resolution is not minimal by Remark 3.2.

Next, we shall see how to determine the maps of this Taylor resolution. To determine φ_1 , we know that φ_1 is a 4×6 matrix, and we construct it by labelling each row with an element $a_j \in S_0$ for $j = 1, 2, 3, 4$ and each column with an element of $b_k \in S_1$ for $k = 1, \dots, 6$. The entries $m_{j,k}$ in the matrix are either

- (i) zero, if b_k is not a least common multiple of a_j .
- (ii) non-zero, with $m_{j,k}$ such that $\pm a_j \cdot m_{j,k} = b_k$ and the sign determined by the orientation of the face.

Repeating this method to obtain all the φ_i for $i = 1, 2, 3$. Hence we have that

$$\begin{aligned} \varphi_1 &= \begin{bmatrix} -s_1 t_1 & -s_0^4 t_0^5 & -s_1^2 t_1^2 & 0 & 0 & 0 \\ s_0 t_0 & 0 & 0 & -s_0^5 t_0^6 & -s_1 t_1 & 0 \\ 0 & s_1^4 t_1^5 & 0 & s_1^5 t_1^6 & 0 & -s_1^6 t_1^7 \\ 0 & 0 & s_0^2 t_0^2 & 0 & s_0 t_0 & s_0^6 t_0^7 \end{bmatrix}, \\ \varphi_2 &= \begin{bmatrix} -s_0^4 t_0^5 & -s_1 t_1 & 0 & 0 \\ s_1 t_1 & 0 & -s_1^2 t_1^2 & 0 \\ 0 & 1 & s_0^4 t_0^5 & 0 \\ -1 & 0 & 0 & s_1 t_1 \\ 0 & -s_0 t_0 & 0 & -s_0^5 t_0^6 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad \varphi_3 = \begin{bmatrix} -s_1 t_1 \\ s_0^4 t_0^5 \\ -1 \\ 1 \end{bmatrix}. \end{aligned}$$

It is obvious that the maps φ_2 and φ_3 contain constants, which confirms that this Taylor resolution is not minimal. In fact, the minimal free resolution is

$$0 \rightarrow S^3 \xrightarrow{\begin{bmatrix} 0 & -s_1 t_1 & 0 & s_0 t_0 \\ -s_1 t_1 & s_0 t_0 & 0 & 0 \\ s_0^4 t_0^5 & 0 & -s_1^4 t_1^5 & 0 \end{bmatrix}^T} S^4 \xrightarrow{[s_0^2 s_1^4 t_0^2 t_1^5, s_0 s_1^5 t_0 t_1^6, s_0^6 t_0^7, s_1^6 t_1^7]} I \rightarrow 0.$$

Example 3.3 shows that the Taylor resolution of the family of monomial ideals I may not be minimal, and the minimal status can be checked

by applying Remark 3.2. It is important to note that the exponents of the monomial generators play a key role in determining if the Taylor's resolution of I is minimal. We shall see later that the ideal I given in Example 3.3 is in case 1 of Corollary 2.4 with $2 = a_0 > a_1 = 1$ and $2 = b_0 > b_1 = 1$. We shall prove that the minimal free resolution of this type of I is $0 \rightarrow S^3 \rightarrow S^4 \rightarrow I \rightarrow 0$ in Theorem 3.7.

In Theorem 3.4 below, we classify all the possible minimal graded free resolutions of this family of generic monomial ideals, and we will prove our theorem by analyzing the exponents of the defining monomials in the three forms described in Corollary 2.4.

Theorem 3.4. *The minimal free resolution of the monomial ideal I given in Theorem 2.2 is in one of the following four forms:*

$$\begin{aligned} 0 \rightarrow S^3 \rightarrow S^4 \rightarrow I \rightarrow 0; \quad 0 \rightarrow S^2 \rightarrow S^5 \rightarrow S^4 \rightarrow I \rightarrow 0; \\ 0 \rightarrow S \rightarrow S^4 \rightarrow S^4 \rightarrow I \rightarrow 0; \quad 0 \rightarrow S \rightarrow S^4 \rightarrow S^6 \rightarrow S^4 \rightarrow I \rightarrow 0. \end{aligned}$$

Proof. Since up to the change of variables, the generic monomial ideal I is in one of the three forms given in Corollary 2.4, we will study the minimal free resolutions of these cases in Theorems 3.5, 3.7, and 3.8 below. The claims here follow directly from Theorems 3.5, 3.7, and 3.8. $\square$

To prove Theorem 3.4, we start with confirming that the Taylor resolution of I is minimal if I is a generic ideal in case 3 of Corollary 2.4. We shall prove this by applying Remark 3.2.

Theorem 3.5. *Let I be a generic ideal in case 3 of Corollary 2.4, where*

$$\begin{aligned} I = (s_0^{a_0} s_1^{m-a_0} t_1^n, s_1^{m-b_1} t_0^{n-b_1}, s_0^{m-b_2} t_1^{n-b_2}, s_0^{a_3} s_1^{m-a_3} t_0^n), \\ 0 < a_0 \neq a_3 < m, 0 < b_1 \neq b_2 < n. \end{aligned}$$

Then the Taylor's resolution is the minimal free resolution. It is of I of the form

$$0 \rightarrow S \rightarrow S^4 \rightarrow S^6 \rightarrow S^4 \rightarrow I \rightarrow 0.$$

Proof. To prove our claim, we only need to check that the condition of Remark 3.2 is satisfied, that is, for all faces $\sigma \in \Delta_I$ and all indices $i \in \sigma$, the monomials m_σ and $m_{\sigma \setminus i}$ are different.

Similar to the construction of Example 3.3, we produce the Taylor's resolution

$$0 \longrightarrow S_3 \xrightarrow{\varphi_3} S_2^4 \xrightarrow{\varphi_2} S_1^6 \xrightarrow{\varphi_1} S_0^4 \longrightarrow I \longrightarrow 0,$$

where the subindex of S_i denote the i -dimensional face, and

$$\begin{aligned} S_0 &= S(s_0^{a_0} s_1^{m-a_0} t_1^n, s_1^{m-b_1} t_0^{n-b_1}, s_0^{m-b_2} t_1^{n-b_2}, s_0^{a_3} s_1^{m-a_3} t_0^n), \\ &\quad 0 < a_0 \neq a_3 < m, 0 < b_1 \neq b_2 < n. \\ &\quad \text{denotes 4 vertices indexed 1, 2, 3, 4} \\ S_1 &= S(s_0^{a_0} s_1^{m-b_1} t_0^n, s_0^{m-a_0} t_0^{b_2} t_1^n, s_0^{\max(a_0, a_3)} s_1^{\max(m-a_0, m-a_3)} t_0^n t_1^n, \\ &\quad s_0^m s_1^{t_0^{\max(b_1, b_2)} t_1^{\max(n-b_1, n-b_2)}}, s_0^{a_3} s_1^{m-b_2} t_0^{n-b_1}, s_0^m s_1^{m-a_3} t_0^n t_1^{n-b_2}) \\ &\quad \text{denotes 6 edges indexed [12], [13], [14], [23], [24], [34]} \\ S_2 &= S(s_0^m s_1^{t_0^{\max(b_1, b_2)} t_1^n}, s_0^{\max(a_0, a_3)} s_1^{m-b_2} t_0^n t_1^n, \\ &\quad s_0^m s_1^{\max(m-a_0, m-a_3)} t_0^n t_1^n, s_0^m s_1^{t_0^{\max(n-b_1, n-b_2)}}) \\ &\quad \text{denotes three 2-dimensional faces indexed [123], [124], [134], [234]} \\ S_3 &= S(s_0^m s_1^{t_0^n t_1^n}). \\ &\quad \text{denotes one 3-dimensional faces indexed [1234].} \end{aligned}$$

We note that since $0 < a_0 \neq a_3 < m$, $0 < b_1 \neq b_2 < n$, the elements of each module are distinct. Hence the Taylor's resolution is the minimal free resolution of I by Remark 3.2. $\square$

Applying the same method, we can verify that if I is the generic ideal in either case 1 or case 2 of Corollary 2.4, then the Taylor resolution of I is not minimal. To construct minimal free resolutions, we first recall the Buchsbaum-Eisenbud exactness criterion for a complex:

Theorem 3.6. (*Buchsbaum-Eisenbud [11]*) *Let S be a commutative Noetherian ring. Let*

$$\mathbb{A} : 0 \longrightarrow F_N \xrightarrow{\phi_N} F_{N-1} \xrightarrow{\phi_{N-1}} F_{N-2} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0$$

be a complex of free finite S -modules. Then $\mathbb{A}$ is exact if and only if for each $i = 1, 2, \dots, N$,

1. $\text{rank } F_i = \text{rank } \phi_i + \text{rank } \phi_{i+1}$;
2. $I(\phi_i)$ contains a S -regular sequence of length i , or $I(\phi_i) = S$, where $I(\phi_i)$ is the ideal generated by the maximal minors of a matrix corresponding to the map ϕ_i .

Next, we will construct minimal free resolution for the generic ideal I in either case 1 or case 2 of Corollary 2.4 of using the Buchsbaum-Eisenbud exactness criterion in Theorem 3.6.

Theorem 3.7. *Let I be a generic ideal in case 1 of Corollary 2.4, where*

$$I = (s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0}, s_0^{a_1} s_1^{m-a_1} t_0^{b_1} t_1^{n-b_1}, s_0^m t_0^n, s_1^m t_1^n),$$

$$0 < a_1 < a_0 < m, 0 < b_0 \neq b_1 < n.$$

Then

(1) if $a_0 > a_1$ and $b_0 > b_1$, then the minimal free resolution of I is $0 \rightarrow S^3 \rightarrow S^4 \rightarrow I \rightarrow 0$;

(2) if $a_0 > a_1$ and $b_0 < b_1$, then the minimal free resolution of I is $0 \rightarrow S^2 \rightarrow S^5 \rightarrow S^4 \rightarrow I \rightarrow 0$.

Proof. (1) If $a_0 > a_1$ and $b_0 > b_1$, then there are six Koszul relations $k_1, \dots, k_6$ of the minimal generators of the ideal I , where $k_1, \dots, k_6$ are given as the columns of the following matrix:

$$[k_1, k_2, k_3, k_4, k_5, k_6]$$

$$= \begin{bmatrix} -s_1^{a_0-a_1} t_1^{b_0-b_1} & -s_0^{m-a_0} t_0^{n-b_0} & -s_1^{a_0} t_1^{b_0} & 0 & 0 & 0 \\ s_0^{a_0-a_1} t_0^{b_0-b_1} & 0 & 0 & -s_0^{m-a_1} t_0^{n-b_1} & -s_1^{a_1} t_1^{b_1} & 0 \\ 0 & s_1^{m-a_0} t_1^{n-b_0} & 0 & s_1^{m-a_1} t_1^{n-b_1} & 0 & -s_1^m t_1^n \\ 0 & 0 & s_0^{a_0} t_0^{b_0} & 0 & s_0^{a_1} t_0^{b_1} & s_0^m t_0^n \end{bmatrix}.$$

Note that k_1, k_3, k_4 are linearly independent over S since at least one of the maximal minor is non-zero, and

$$\begin{aligned} k_3 &= s_1^{a_1} t_1^{b_1} k_1 + s_0^{a_0-a_1} t_0^{b_0-b_1} k_5; \\ k_4 &= -s_0^{m-a_0} t_0^{n-b_0} k_1 + s_1^{a_0-a_1} t_1^{b_0-b_1} k_2; \\ k_6 &= -s_1^{a_0} t_1^{b_0} k_2 + s_0^{m-a_0} t_0^{n-b_0} k_3 \\ &= -s_1^{a_0} t_1^{b_0} k_2 + s_0^{m-a_0} t_0^{n-b_0} (s_1^{a_1} t_1^{b_1} k_1 + s_0^{a_0-a_1} t_0^{b_0-b_1} k_5). \end{aligned}$$

Therefore, the Koszul syzygies are generated by k_1 , k_2 , and k_5 . Furthermore, the outer product (the maximal minors with alternating signs) of the matrix

$$\{k_1, k_2, k_5\}_o = \begin{Bmatrix} -s_1^{a_0-a_1}t_1^{b_0-b_1} & -s_0^{m-a_0}t_0^{n-b_0} & 0 \\ s_0^{a_0-a_1}t_0^{b_0-b_1} & 0 & -s_1^{a_1}t_1^{b_1} \\ 0 & s_1^{m-a_0}t_1^{n-b_0} & 0 \\ 0 & 0 & s_0^{a_1}t_0^{b_1} \end{Bmatrix}_o$$

$$= \begin{bmatrix} s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0} & s_0^{a_1} s_1^{m-a_1} t_0^{b_1} t_1^{n-b_1} & s_0^m t_0^n & s_1^m t_1^n \end{bmatrix}.$$

Thus, if $a_0 > a_1$ and $b_0 > b_1$, then, by the Hilbert-Burch theorem, the minimal free resolution of I is of the form

$$0 \longrightarrow S^3 \xrightarrow{\phi_1=[k_1, k_2, k_5]} S^4 \longrightarrow I \longrightarrow 0.$$

(2) If $a_0 > a_1$ and $b_0 < b_1$, then there are six Koszul relations $k_1, \dots, k_6$ of the minimal generators of the ideal I , where $k_1, \dots, k_6$ are given as the columns of the following matrix:

$$[k_1, k_2, k_3, k_4, k_5, k_6]$$

$$= \begin{bmatrix} -s_1^{a_0-a_1}t_0^{b_1-b_0} & -s_0^{m-a_0}t_0^{n-b_0} & -s_1^{a_0}t_1^{b_0} & 0 & 0 & 0 \\ s_0^{a_0-a_1}t_1^{b_1-b_0} & 0 & 0 & -s_0^{m-a_1}t_0^{n-b_1} & -s_1^{a_1}t_1^{b_1} & 0 \\ 0 & s_1^{m-a_0}t_1^{n-b_0} & 0 & s_1^{m-a_1}t_1^{n-b_1} & 0 & -s_1^m t_1^n \\ 0 & 0 & s_0^{a_0}t_0^{b_0} & 0 & s_0^{a_1}t_0^{b_1} & s_0^m t_0^n \end{bmatrix}.$$

Note that

$$\begin{aligned} k_6 &= -s_1^{a_1}t_1^{b_1}k_4 + s_0^{m-a_1}t_0^{n-b_1}k_5; \\ 0 &= -s_0^{m-a_0}t_0^{n-b_1}k_1 + s_1^{a_0-a_1}k_2 - t_1^{b_1-b_0}k_4; \\ 0 &= -s_1^{a_1}t_1^{b_0}k_1 + t_0^{b_1-b_0}k_3 - s_0^{a_0-a_1}k_5. \end{aligned}$$

Therefore, the Koszul syzygies are generated by k_1, k_2, k_3, k_4, k_5 ; and the second syzygies are generated by

$$[g_1, g_2] = \begin{bmatrix} -s_0^{m-a_0}t_0^{n-b_1} & -s_1^{a_1}t_1^{b_0} \\ s_1^{a_0-a_1} & 0 \\ 0 & t_0^{b_1-b_0} \\ -t_1^{b_1-b_0} & 0 \\ 0 & -s_0^{a_0-a_1} \end{bmatrix}, \text{ such that } [k_1, k_2, k_3, k_4, k_5] \cdot [g_1, g_2] = 0.$$

Note, the determinant of any 4×4 minor of the matrix $[k_1, k_2, k_3, k_4, k_5]$ is zero, and it is easy to observe that the determinant of the 3×3 minor formed by the last three rows of $[k_1, k_2, k_3]$ is $s_0^{2a_0-a_1} s_1^{m-a_0} t_0^{b_0} t_1^{n+b_1-2b_0} \neq 0$. Thus $\text{rank}[k_1, k_2, k_3, k_4, k_5] = 3$; and the depth of the Fitting ideal is at least one, since $s_0^{2a_0-a_1} s_1^{m-a_0} t_0^{b_0} t_1^{n+b_1-2b_0}$ is not a zero-divisor in S . In addition, $\text{rank}[g_1, g_2] = 2$, and the depth of the Fitting ideal is at least two, since $(s_0 s_1)^{a_0-a_1}$ and $(t_0 t_1)^{b_1-b_0}$ are two maximal minors, which is regular sequence of length 2 in S .

Therefore, if $a_0 > a_1$ and $b_0 < b_1$, then, by the Buchsbaum-Eisenbud exactness criterion in Theorem 3.6, the minimal free resolution of I is of the form

$$0 \rightarrow S^2 \xrightarrow{\phi_2=[g_1, g_2]} S^5 \xrightarrow{\phi_1=[k_1, k_2, k_3, k_4, k_5]} S^4 \longrightarrow I \rightarrow 0.$$

□

Applying similar technique, we will finalize the minimal free resolution for a generic ideal I in case 2 of Corollary 2.4.

Theorem 3.8. *Let I be a generic ideal in case 2 of Corollary 2.4, where*

$$I = (s_0^{a_0} s_1^{m-a_0} t_0^{b_0} t_1^{n-b_0}, s_1^{m-b_1} t_0^{b_1} t_1^{n-b_1}, s_0^m t_0^n, s_0^{a_3} s_1^{m-a_3} t_1^n),$$

$$0 < a_0 \neq a_3 < m, 0 < b_0 \neq b_1 < n.$$

Then

(1) if $a_0 > a_3$ and $b_0 > b_1$, then the minimal free resolution of I is $0 \rightarrow S \rightarrow S^4 \rightarrow S^4 \rightarrow I \rightarrow 0$;

(2) if $a_0 > a_3$ and $b_0 < b_1$, or $a_0 < a_3$ and $b_0 > b_1$, or $a_0 < a_3$ and $b_0 < b_1$, then the minimal free resolution of I is $0 \rightarrow S^2 \rightarrow S^5 \rightarrow S^4 \rightarrow I \rightarrow 0$.

Proof. (1) If $a_0 > a_3$ and $b_0 > b_1$, then there are six Koszul relations $k_1, \dots, k_6$ of the minimal generators of the ideal I , where $k_1, \dots, k_6$ are given as the columns of the following matrix:

$$= \begin{bmatrix} [k_1, k_2, k_3, k_4, k_5, k_6] \\ \begin{matrix} -s_1^{a_0} t_1^{b_0-b_1} & -s_0^{m-a_0} t_0^{n-b_0} & -s_1^{a_0-a_3} t_1^{b_0} & 0 & 0 & 0 \\ s_0^{a_0} t_0^{b_0-b_1} & 0 & 0 & -s_0^m t_0^{n-b_1} & -s_0^{a_3} t_1^{b_1} & 0 \\ 0 & s_1^{m-a_0} t_1^{n-b_0} & 0 & s_1^m t_1^{n-b_1} & 0 & -s_1^{m-a_3} t_1^n \\ 0 & 0 & s_0^{a_0-a_3} t_0^{b_0} & 0 & s_1^{a_3} t_0^{b_1} & s_0^{m-a_3} t_0^n \end{matrix} \end{bmatrix}.$$

In addition, we see that

$$\begin{aligned} k_4 &= -s_0^{m-a_0} t_0^{n-b_0} k_1 + s_1^{a_0} t_1^{b_0-b_1} k_2, \\ k_6 &= -s_1^{a_0-a_3} t_1^{b_0} k_2 + s_0^{m-a_0} t_0^{n-b_0} k_3 \\ 0 &= -t_1^{b_1} k_1 + s_1^{a_3} k_3 - s_0^{a_0-a_3} t_0^{b_0-b_1} k_5. \end{aligned}$$

Therefore, the Koszul syzygies are generated by k_1, k_2, k_3, k_5 ; and the second syzygies are generated by

$$g_1 = [-t_1^{b_1} \ 0 \ s_1^{a_3} \ -s_0^{a_0-a_3} t_0^{b_0-b_1}]^T \text{ such that } [k_1, k_2, k_3, k_5] \cdot g_1 = 0.$$

We observe that the determinant of the matrix $[k_1, k_2, k_3, k_5]$ is zero due to the relation g_1 among the columns, and the determinant of a 3×3 minor formed by the last three rows of $[k_1, k_2, k_3]$ is $s_0^{2a_0-a_3} s_1^{m-a_0} t_0^{2b_0-b_1} t_1^{n-b_0} \neq 0$. Thus $\text{rank}[k_1, k_2, k_3, k_5] = 3$; and the depth of the Fitting ideal is at least one, since $s_0^{2a_0-a_3} s_1^{m-a_0} t_0^{2b_0-b_1} t_1^{n-b_0}$ is not a zero-divisor in S . In addition, $\text{rank}(g_1) = 1$, and the depth of the Fitting ideal is at least two, since $-t_1^{b_1}$ and $s_1^{a_3}$ are two maximal minors that form a regular sequence of length 2 in S .

Thus, if $a_0 > a_3$ and $b_0 > b_1$, then, by the Buchsbaum-Eisenbud exactness criterion in Theorem 3.6, the minimal free resolution of I is of the form

$$0 \longrightarrow S \xrightarrow{\phi_2=g_1} S^4 \xrightarrow{\phi_1=[k_1, k_2, k_3, k_5]} S^4 \longrightarrow I \longrightarrow 0.$$

(2a) If $a_0 > a_3$ and $b_0 < b_1$, then there are six Koszul relations $k_1, \dots, k_6$ of the minimal generators of the ideal I , where $k_1, \dots, k_6$ are given as the columns of the following matrix:

$$[k_1, k_2, k_3, k_4, k_5, k_6] = \begin{bmatrix} -s_1^{a_0} t_0^{b_1-b_0} & -s_0^{m-a_0} t_0^{n-b_0} & -s_1^{a_0-a_3} t_1^{b_0} & 0 & 0 & 0 \\ s_0^{a_0} t_1^{b_1-b_0} & 0 & 0 & -s_0^m t_0^{n-b_1} & -s_0^{a_3} t_1^{b_1} & 0 \\ 0 & s_1^{m-a_0} t_1^{n-b_0} & 0 & s_1^m t_1^{n-b_1} & 0 & -s_1^{m-a_3} t_1^n \\ 0 & 0 & s_0^{a_0-a_3} t_0^{b_0} & 0 & s_1^{a_3} t_0^{b_1} & s_0^{m-a_3} t_0^n \end{bmatrix}.$$

In addition, we see that

$$\begin{aligned} k_6 &= -s_1^{a_0-a_3} t_1^{b_0} k_2 + s_0^{m-a_0} t_0^{n-b_0} k_3; \\ 0 &= -t_1^{b_0} k_1 + s_1^{a_3} t_0^{b_1-b_0} k_3 - s_0^{a_0-a_3} k_5; \\ 0 &= -s_0^{m-a_0} t_0^{n-b_1} k_1 + s_1^{a_0} k_2 - t_1^{b_1-b_0} k_4. \end{aligned}$$

Therefore, the Koszul syzygies are generated by k_1, k_2, k_3, k_4, k_5 ; and the second syzygies are generated by

$$[g_1, g_2] = \begin{bmatrix} -t_1^{b_0} & -s_0^{m-a_0}t_0^{n-b_1} \\ 0 & s_1^{a_0} \\ s_1^{a_3}t_0^{b_1-b_0} & 0 \\ 0 & -t_1^{b_1-b_0} \\ -s_0^{a_0-a_3} & 0 \end{bmatrix}, \text{ such that } [k_1, k_2, k_3, k_4, k_5] \cdot [g_1, g_2] = \mathbf{0}.$$

Note, the determinant of any 4×4 minor of the matrix $[k_1, k_2, k_3, k_4, k_5]$ is zero, and the determinant of a 3×3 minor formed by the last three rows of $[k_1, k_2, k_3]$ is $s_0^{2a_0-a_3}s_1^{m-a_0}t_0^{b_0}t_1^{n+b_1-2b_0} \neq 0$. Thus $\text{rank}[k_1, k_2, k_3, k_4, k_5] = 3$; and the depth of the Fitting ideal is at least one, since $s_0^{2a_0-a_3}s_1^{m-a_0}t_0^{b_0}t_1^{n+b_1-2b_0}$ is not a zero-divisor in S . In addition, $\text{rank}[g_1, g_2] = 2$, and the depth of the Fitting ideal is at least two, since $s_1^{a_0+a_3}t_0^{b_1-b_0}$ and $t_1^{b_1}$ are two maximal minors that form a regular sequence of length 2 in S .

Therefore, if $a_0 > a_3$ and $b_0 < b_1$, then, by the Buchsbaum-Eisenbud exactness criterion in Theorem 3.6, the minimal free resolution of I is of the form

$$0 \rightarrow S^2 \xrightarrow{\phi_2=[g_1, g_2]} S^5 \xrightarrow{\phi_1=[k_1, k_2, k_3, k_4, k_5]} S^4 \longrightarrow I \rightarrow 0. \quad (3)$$

Similar arguments can show the other cases. For simplicity, we only give the maps of the free resolutions below.

(2b) If $a_0 < a_3$ and $b_0 > b_1$, then the minimal free resolution of I is of the form in (3), where

$$\begin{aligned} \phi_1 &= \begin{bmatrix} -s_1^{a_0}t_1^{b_0-b_1} & -s_0^{m-a_0}t_0^{n-b_0} & -s_0^{a_3-a_0}t_1^{b_0} & 0 & 0 \\ s_0^{a_0}t_0^{b_0-b_1} & 0 & 0 & -s_0^{a_3}t_1^{b_1} & 0 \\ 0 & s_1^{m-a_0}t_1^{n-b_0} & 0 & 0 & -s_1^{m-a_3}t_1^n \\ 0 & 0 & s_1^{a_3-a_0}t_0^{b_0} & s_1^{a_3}t_0^{b_1} & s_0^{m-a_3}t_0^n \end{bmatrix}, \\ \phi_2 &= \begin{bmatrix} -s_0^{a_3-a_0}t_1^{b_1} & 0 \\ 0 & -t_1^{b_0} \\ s_1^{a_0} & s_0^{m-a_3}t_0^{n-b_0} \\ -t_0^{b_0-b_1} & 0 \\ 0 & -s_1^{a_3-a_0} \end{bmatrix}. \end{aligned}$$

(2c) If $a_0 < a_3$ and $b_0 < b_1$, then the minimal free resolution of I is of the form in (3), where

$$\begin{aligned} \phi_1 &= \begin{bmatrix} -s_1^{a_0} t_0^{b_1-b_0} & -s_0^{m-a_0} t_0^{n-b_0} & -s_0^{a_3-a_0} t_1^{b_0} & 0 & 0 \\ s_0^{a_0} t_1^{b_1-b_0} & 0 & 0 & -s_0^m t_0^{n-b_1} & 0 \\ 0 & s_1^{m-a_0} t_1^{n-b_0} & 0 & s_1^m t_1^{n-b_1} & -s_1^{m-a_3} t_1^n \\ 0 & 0 & s_1^{a_3-a_0} t_0^{b_0} & 0 & s_0^{m-a_3} t_0^n \end{bmatrix}, \\ \phi_2 &= \begin{bmatrix} -s_0^{m-a_0} t_0^{n-b_1} & 0 \\ s_1^{a_0} & -t_1^{b_0} \\ 0 & s_0^{m-a_3} t_0^{n-b_0} \\ -t_1^{b_1-b_0} & 0 \\ 0 & -s_1^{a_3-a_0} \end{bmatrix}. \end{aligned}$$

□

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**HAMILTON-JACOBI QUANTIZATION OF
LANDAU-GINZBURG THEORY**

Walaa I. Eshraim

New York University Abu Dhabi
Saadiyat Island, P.O. Box 129188
Abu Dhabi, U.A.E.
wie2003@nyu.edu

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Abstract

We discuss the Hamilton-Jacobi approach for a constrained system. We obtain the equation of motion for a singular systems as total differential equations in many variables. We investigate the integrability conditions without using any gauge fixing condition. The path integral quantization for systems with finite degrees of freedom is applied to the field theories with constraints. So, we apply the Hamilton-Jacobi quantization to obtain the path integral of the Landau-Ginzburg theory.

keywords: Hamilton-Jacobi formalism, Singular Lagrangian, Path integral quantization of constrained systems.

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1 Introduction

The quantization of constrained Hamiltonian systems can be achieved by means of operator methods [1, 2] or by path integral quantization [3, 4, 5].

The quantization of a massive spin one particle became the center of interest of physicists especially after the pioneering work of Faddeev [3] who introduced the path integral quantization of singular theories which possess first-class constraints in canonical gauge. Since first class constraints are generators of gauge transformations, this will lead to the gauge freedom. In other words, the equations of motion are still degenerate and depend on the functional arbitrariness, one has to impose external gauge constraints for each first-class constraint.

Using Dirac's method, one can see that the gauge fixing is not always an easy task [6, 7, 8]. But, if the canonical method or Hamilton-Jacobi approach [9, 10] is used, the gauge fixing is not necessary to analyze the singular systems [11]. The Hamilton-Jacobi treatment of constrained systems leads us to obtain the equations of motion as total differential equations in many variables, as seen in Refs. [12, 13]. By using Hamilton-Jacobi quantization method, the Path integral quantization has been obtained for several constraint systems: (i) the Scalar field coupled minimally to the vector potential [14], (ii) the electromagnetic field coupled to a spinor [15], (iii) the relativistic local free field theory [13], (iv) the scalar field coupled to two flavours Fermionic through Yukawa couplings [13].

The aim of this paper is to treat the Landau-Ginzburg theory that gives an effective description of phenomenon precisely coincides with scalar quantum electrodynamics as constrained system. The path integral quantization is obtained by using canonical method.

2 Hamilton-Jacobi Formalism Of Constrained Systems

In this section, we study the constrained systems by using the canonical

method [9, 10] and demonstrate the fact that the gauge fixing problem is solved naturally. The starting point of this method is to consider the lagrangian $L \equiv L(q_i, \dot{q}_i, \tau)$, $i = 1, 2, \dots, n$ with Hessian matrix

$$A_{ij} = \frac{\partial^2 L(q_i, \dot{q}_i, \tau)}{\partial \dot{q}_i \partial \dot{q}_j}, \quad i, j = 1, 2, \dots, n, \quad (1)$$

of rank $(n - r)$, $r < n$. Then r momenta are dependent. The generalized momenta p_i corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, 2, \dots, n - r, \quad (2)$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \quad \mu = n - r + 1, \dots, n. \quad (3)$$

enable us to solve Eq. (2) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a(q_i, p_a, \dot{q}_\mu; \tau) \equiv \omega_a. \quad (4)$$

Substituting Eq. (4), into Eq. (3), we get

$$p_\mu = \left. \frac{\partial L}{\partial \dot{q}_\mu} \right|_{\dot{q}_a = \omega_a} \equiv -H_\mu(q_i, \dot{q}_\mu, p_a; t). \quad (5)$$

Relations (5) indicate the fact that the generalized momenta p_μ are not independent of p_a which is a natural result of the singular nature of the lagrangian.

The canonical Hamiltonian H_0 is defined as

$$H_0 = -L(q_i, \dot{q}_\mu, \dot{q}_a \equiv \omega_a; \tau) + p_a \dot{q}_a + p_\mu \dot{q}_\mu \Big|_{p_\mu = -H_\mu}. \quad (6)$$

The set of Hamilton-Jacobi Partial Differential Equations is expressed as

$$H'_\alpha \left(\tau, q_\mu, q_a, P_i = \frac{\partial S}{\partial q_i}, P_0 = \frac{\partial S}{\partial \tau} \right) = 0, \quad \alpha = 0, n - p + 1, \dots, n, \quad (7)$$

where

$$H'_\alpha = p_\alpha + H_\alpha. \quad (8)$$

The equations of motion are obtained as total differential equation in many variables as follows:

$$dq_r = \frac{\partial H'_\alpha}{\partial p_r} dt_\alpha, \quad r = 0, 1, \dots, n, \quad (9)$$

$$dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \quad a = 1, \dots, n-p, \quad (10)$$

$$dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu} dt_\alpha, \quad \alpha = 0, n-p+1, \dots, n, \quad (11)$$

$$dz = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha. \quad (12)$$

where $Z = S(t_\alpha, q_a)$. The set of equations (9-12) are integrable if

$$dH'_\alpha = 0, \quad \alpha = 0, n-p+1, \dots, n. \quad (13)$$

If conditions (13) are not satisfied identically, one may consider them as new constraint and a gain test the integrability conditions, then repeating this procedure, a set of conditions may be obtained.

In this case of integrable system, the path integral representation may be written as [16, 17, 18, 19].

$$\langle Out | S | In \rangle = \int \prod_{a=1}^{n-r} dq^a dp^a \exp \left[i \int_{t_\alpha}^{t'_\alpha} \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha \right]. \quad (14)$$

One should notice that the integral (14) is an integration over the canonical phase space coordinates q_a, p_a .

3 The Landau-Ginzburg theory

The Landau-Ginzburg theory that gives an effective description of phenomenon precisely coincides with scalar quantum electrodynamics is described by the lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \varphi)^* D^\mu \varphi - k \varphi^* \varphi - \frac{1}{4} \lambda (\varphi^* \varphi)^2, \quad (15)$$

where

$$D_\mu \varphi = \partial_\mu \varphi - ie A_\mu \varphi. \quad (16)$$

In the Landau-Ginzburg theory φ describes the Cooper pairs. In usual quantum electrodynamics, we would put $k = m^2$, where m is the effective mass of electron.

The Lagrangian function (15) is singular, since the rank of the Hessian matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}. \quad (17)$$

is three.

The canonical momenta are defined as

$$\pi^i = \frac{\partial L}{\partial \dot{A}_i} = -F^{0i}, \quad (18)$$

$$\pi^0 = \frac{\partial L}{\partial \dot{A}_0} = 0, \quad (19)$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = (D_0 \varphi)^* = \dot{\varphi}^* + ie A_0 \varphi^*, \quad (20)$$

$$p_{\varphi^*} = \frac{\partial L}{\partial \dot{\varphi}^*} = (D_0 \varphi) = \dot{\varphi} - ie A_0 \varphi, \quad (21)$$

From Eqs. (18), (20) and (21), the velocities $\dot{A}_i, \dot{\varphi}^*$ and $\dot{\varphi}$ can be expressed in terms of momenta π_i, p_φ and p_{φ^*} respectively as

$$\dot{A}_i = -\pi_i - \partial_i A_0, \quad (22)$$

$$\dot{\varphi}^* = p_\varphi - ie A_0 \varphi^*, \quad (23)$$

$$\dot{\varphi} = p_{\varphi^*} + ie A_0 \varphi. \quad (24)$$

The canonical Hamiltonian H_0 is obtained as

$$\begin{aligned} H_0 = & \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi_i \pi^i + \pi^i \partial_i A_0 + p_{\varphi^*} p_\varphi + ie A_0 \varphi p_\varphi \\ & - ie A_0 \varphi^* p_{\varphi^*} - (D_i \varphi)^* (D^i \varphi) + k \varphi^* \varphi + \frac{1}{4} \lambda (\varphi^* \varphi)^2. \end{aligned} \quad (25)$$

Making use of (7) and (8), we find for the set of HJPDE

$$H'_0 = \pi_4 + H_0, \quad (26)$$

$$H' = \pi_0 + H = \pi_0 = 0, \quad (27)$$

Therefor, the total differential equations for the characteristic (9-11) obtained as

$$\begin{aligned} dA^i &= \frac{\partial H'_0}{\partial \pi_i} dt + \frac{\partial H'}{\partial \pi_i} dA^0, \\ &= -(\pi^i + \partial_i A_0) dt, \end{aligned} \quad (28)$$

$$dA^0 = \frac{\partial H'_0}{\partial \pi_0} dt + \frac{\partial H'}{\partial \pi_0} dA^0 = dA^0, \quad (29)$$

$$\begin{aligned} d\varphi &= \frac{\partial H'_0}{\partial p_\varphi} dt + \frac{\partial H'}{\partial p_\varphi} dA^0, \\ &= (p_{\varphi^*} + ieA_0\varphi) dt, \end{aligned} \quad (30)$$

$$\begin{aligned} d\varphi^* &= \frac{\partial H'_0}{\partial p_{\varphi^*}} dt + \frac{\partial H'}{\partial p_{\varphi^*}} dA^0, \\ &= (p_\varphi - ieA_0\varphi^*) dt, \end{aligned} \quad (31)$$

$$\begin{aligned} d\pi^i &= -\frac{\partial H'_0}{\partial A_i} dt - \frac{\partial H'}{\partial A_i} dA^0, \\ &= [\partial_i F^{li} + ie(\varphi^* \partial^i \varphi + \varphi \partial_i \varphi^*) + 2e^2 A^i \varphi \varphi^*] dt, \end{aligned} \quad (32)$$

$$\begin{aligned} d\pi^0 &= -\frac{\partial H'_0}{\partial A_0} dt - \frac{\partial H'}{\partial A_0} dA^0, \\ &= [\partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi p_\varphi] dt, \end{aligned} \quad (33)$$

$$\begin{aligned} dp_\varphi &= -\frac{\partial H'_0}{\partial \varphi} dt - \frac{\partial H'}{\partial \varphi} dA^0, \\ &= [(\vec{D} \cdot \vec{D}\varphi)^* - k\varphi^* - \frac{1}{2}\lambda\varphi\varphi^{*2} - ieA_0p_\varphi] dt, \end{aligned} \quad (34)$$

and

$$\begin{aligned} dp_{\varphi^*} &= -\frac{\partial H'_0}{\partial \varphi^*} dt - \frac{\partial H'}{\partial \varphi^*} dA^0, \\ &= [(\vec{D} \cdot \vec{D}\varphi) - k\varphi - \frac{1}{2}\lambda\varphi^*\varphi^2 + ieA_0p_{\varphi^*}] dt. \end{aligned} \quad (35)$$

The integrability condition ($dH'_\alpha = 0$) implies that the variation of the constraint H' should be identically zero, that is

$$dH' = d\pi_0 = 0, \quad (36)$$

which leads to a new constraint

$$H'' = \partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi p_\varphi = 0. \quad (37)$$

Taking the total differential of H'' , we have

$$dH'' = \partial_i d\pi^i + iep_{\varphi^*} d\varphi^* + ie\varphi^* dp_{\varphi^*} - ie\varphi dp_\varphi - iep_\varphi d\varphi = 0. \quad (38)$$

Then the set of equation (28-35) is integrable. Therefore, the canonical phase space coordinates (φ, p_φ) and $(\varphi^*, p_{\varphi^*})$ are obtained in terms of parameters (t, A^0) .

Making use of Eq.(12) and (25-27), we obtain the canonical action integral as

$$Z = \int d^4x \left(-\frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi_i \pi^i + p_\varphi p_{\varphi^*} + \vec{D}\varphi^* \cdot \vec{D}\varphi + k|\varphi|^2 + \frac{1}{4} \lambda (\varphi^* \varphi)^2 \right), \quad (39)$$

where

$$\vec{D} = \vec{\nabla} + ie\vec{A}. \quad (40)$$

Now the path integral representation (14) is given by

$$\langle out|S|In\rangle = \int \prod_i dA^i d\pi^i d\varphi dp_\varphi d\varphi^* dp_{\varphi^*} \exp \left[i \left\{ \int d^4x \left(-\frac{1}{2} \pi_i \pi^i - \frac{1}{4} F^{ij} F_{ij} + p_\varphi p_{\varphi^*} + (D_i \varphi)^* (D_i \varphi) - k \varphi^* \varphi - \frac{1}{4} \lambda (\varphi^* \varphi)^2 \right) \right\} \right]. \quad (41)$$

4 Conclusion

In this paper, the Landau-Ginzburg theory gives an effective description of phenomenon precisely coincides with scalar quantum electrodynamics. The Landau-Ginzburg has been quantized by constructing a path integral quantization within the canonical method to constrained system. The equations of motion are obtained as total differential equations in many variables. All the constraints coming from the Hamiltonian procedure and the integrability conditions have been derived. The path integral quantization is performed by using the action which is given by canonical method. The integration is taken over the canonical phase space.

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**ARBITRARY LORENTZ TRANSFORMATION: ITS FACTORIZATION
IN TERMS OF CAYLEY-KLEIN PARAMETERS**

A. Hernández-Galeana, J. Rivera-Rebolledo

Departamento de Física, ESFM, Instituto Politécnico Nacional
Col. Lindavista CP 07738 CDMX, México
ahernandez@ipn.mx; jrivera@esfm.ipn.mx
and

J. López-Bonilla, S. Vidal-Beltrán

ESIME-Zacatenco, Instituto Politécnico Nacional
Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México
jlopezb@ipn.mx; svidalb@ipn.mx

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Abstract

The Olinde-Cartan's expression allows to deduce a factorization for an arbitrary Lorentz mapping, which is useful to obtain the generating orthogonal matrix of 3-rotations in terms of Euler angles and the matrix of transformation for a 4-spinor of Dirac under a Lorentz matrix.

Keywords: Lorentz matrix, Olinde-Cartan's relation, Special relativity, Infeld-van der Waerden's symbols, Null tetrad, Unimodular matrix.

The Olinde Rodrigues [1]-Cartan [2] expression:

$$\begin{pmatrix} \tilde{x}^0 + \tilde{x}^3 & \tilde{x}^1 + i \tilde{x}^2 \\ \tilde{x}^1 - i \tilde{x}^2 & \tilde{x}^0 - \tilde{x}^3 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x^0 + x^3 & x^1 + i x^2 \\ x^1 - i x^2 & x^0 - x^3 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}, \quad (1)$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary complex numbers verifying the condition $\alpha\delta - \beta\gamma = 1$, implies six degrees of freedom for the Lorentz matrix $L = (L^\nu_\mu)$ between the frames of reference $(x^\mu) = (ct, x, y, z)$ and $(\tilde{x}^\nu)$:

$$\tilde{x}^\nu = L^\nu_\mu x^\mu. \quad (2)$$

From (1) and (2) we obtain the relations [3-9]:

$$\begin{aligned} L^0_0 &= \frac{1}{2}(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}), & L^1_0 &= \frac{1}{2}(\bar{\alpha}\gamma + \bar{\beta}\delta) + cc, \\ L^2_0 &= -\frac{i}{2}(\alpha\bar{\gamma} - \bar{\beta}\delta) + cc, \\ L^0_1 &= \frac{1}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, & L^1_1 &= \frac{1}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, \\ L^2_1 &= -\frac{i}{2}(\alpha\bar{\delta} + \beta\bar{\gamma}) + cc, \\ L^0_2 &= -\frac{i}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, & L^1_2 &= -\frac{i}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, \\ L^2_2 &= \frac{1}{2}(\bar{\alpha}\delta - \bar{\beta}\gamma) + cc, \\ L^0_3 &= \frac{1}{2}(\alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma} - \delta\bar{\delta}), & L^1_3 &= \frac{1}{2}(\bar{\alpha}\gamma - \bar{\beta}\delta) + cc, \\ L^2_3 &= -\frac{i}{2}(\alpha\bar{\gamma} + \bar{\beta}\delta) + cc, \\ L^3_0 &= \frac{1}{2}(\alpha\bar{\alpha} + \beta\bar{\beta} - \gamma\bar{\gamma} - \delta\bar{\delta}), & L^3_1 &= \frac{1}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc, \\ L^3_2 &= -\frac{i}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc, \end{aligned} \quad (3)$$

$$L^3_3 = \frac{1}{2}(\alpha\bar{\alpha} - \beta\bar{\beta} - \gamma\bar{\gamma} + \delta\bar{\delta}), \quad \alpha\delta - \beta\gamma = 1,$$

where cc means the complex conjugate of all the previous terms.

An analysis of the expressions (3) shows that the corresponding Lorentz matrix admits the following factorization:

$$(L^\nu_\mu) = \frac{1}{2} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \gamma & \delta & \alpha & \beta \\ i\gamma & i\delta & -i\alpha & -i\beta \\ \alpha & \beta & -\gamma & -\delta \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} & i\bar{\beta} & \bar{\alpha} \\ \bar{\beta} & \bar{\alpha} & -i\bar{\alpha} & -\bar{\beta} \\ \bar{\gamma} & \bar{\delta} & i\bar{\delta} & \bar{\gamma} \\ \bar{\delta} & \bar{\gamma} & -i\bar{\gamma} & -\bar{\delta} \end{pmatrix}, \quad (4)$$

and we consider that it is interesting to analyze the possible physical meaning of this splitting (4).

If (1) is written in the form:

$$\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \begin{pmatrix} \tilde{x}^0 + \tilde{x}^3 & \tilde{x}^1 + i\tilde{x}^2 \\ \tilde{x}^1 - i\tilde{x}^2 & \tilde{x}^0 - \tilde{x}^3 \end{pmatrix} = \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}, \quad (5)$$

then (4) is immediate from (2) and (5). The inversion of (4) implies the relation:

$$L^{-1} = \frac{1}{2} \begin{pmatrix} \bar{\delta} & -\bar{\gamma} & -\bar{\beta} & \bar{\alpha} \\ -\bar{\gamma} & \bar{\delta} & \bar{\alpha} & -\bar{\beta} \\ i\bar{\gamma} & i\bar{\delta} & -i\bar{\alpha} & -i\bar{\beta} \\ \bar{\delta} & \bar{\gamma} & -\bar{\beta} & -\bar{\alpha} \end{pmatrix} \begin{pmatrix} \delta & -\beta & i\beta & \delta \\ -\gamma & \alpha & -i\alpha & -\gamma \\ -\beta & \delta & i\delta & \beta \\ \alpha & -\gamma & -i\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} L^0_0 & -L^1_0 & -L^2_0 & -L^3_0 \\ -L^0_1 & L^1_1 & L^2_1 & L^3_1 \\ -L^0_2 & L^1_2 & L^2_2 & L^3_2 \\ -L^0_3 & L^1_3 & L^2_3 & L^3_3 \end{pmatrix}, \quad (6)$$

in agreement with the known result [5]:

$$L^{-1}{}^a{}_b = \eta_{(b)(c)} L^c{}_r \eta^{(r)(a)}. \quad (7)$$

The unimodular matrix $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ generates to L , therefore L^{-1} is generated by $B^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$, then if in (3) we make the changes $\alpha \rightarrow \delta, \beta \rightarrow -\beta, \gamma \rightarrow -\gamma, \delta \rightarrow \alpha$ we obtain the properties:

$$L^{-1}{}^0{}_0 = L^0{}_0, \quad L^{-1}{}^0{}_j = -L^j{}_0, \quad L^{-1}{}^j{}_0 = -L^0{}_j, \quad L^{-1}{}^j{}_k = L^k{}_j, \quad (8)$$

which is equivalent to (6).

In Minkowski spacetime, two real orthonormal tetrads are connected by a Lorentz transformation [10]:

$$\tilde{e}^{(a)}{}_\mu = L^a{}_b e^{(b)}{}_\mu, \quad (9)$$

similar to (2), then (1) is applicable:

$$B \begin{pmatrix} \tilde{e}^{(0)}{}_\mu + \tilde{e}^{(3)}{}_\mu & \tilde{e}^{(1)}{}_\mu + i \tilde{e}^{(2)}{}_\mu \\ \tilde{e}^{(1)}{}_\mu - i \tilde{e}^{(2)}{}_\mu & \tilde{e}^{(0)}{}_\mu - \tilde{e}^{(3)}{}_\mu \end{pmatrix} = B \begin{pmatrix} e^{(0)}{}_\mu + e^{(3)}{}_\mu & e^{(1)}{}_\mu + i e^{(2)}{}_\mu \\ e^{(1)}{}_\mu - i e^{(2)}{}_\mu & e^{(0)}{}_\mu - e^{(3)}{}_\mu \end{pmatrix} B^\dagger, \quad (10)$$

or in terms of the Newman-Penrose's tetrad [6, 11-13]:

$$\begin{pmatrix} \tilde{n}_\mu & -\tilde{\bar{m}}_\mu \\ -\tilde{m}_\mu & \tilde{l}_\mu \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} n_\mu & -\bar{m}_\mu \\ -m_\mu & l_\mu \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}, \quad (11)$$

which implies the following relations of transformation for the null tetrad [14, 15]:

$$\tilde{l}^\mu = \delta \bar{\delta} l^\mu + \gamma \bar{\gamma} n^\mu - \delta \bar{\gamma} m^\mu - \gamma \bar{\delta} \bar{m}^\mu,$$

$$\begin{aligned}
 \tilde{n}^\mu &= \beta \bar{\beta} l^\mu + \alpha \bar{\alpha} n^\mu - \beta \bar{\alpha} m^\mu - \alpha \bar{\beta} \bar{m}^\mu, \\
 \tilde{m}^\mu &= -\delta \bar{\beta} l^\mu - \gamma \bar{\alpha} n^\mu + \delta \bar{\alpha} m^\mu + \gamma \bar{\beta} \bar{m}^\mu, \\
 \tilde{\bar{m}}^\mu &= -\beta \bar{\delta} l^\mu - \alpha \bar{\gamma} n^\mu + \beta \bar{\gamma} m^\mu + \alpha \bar{\delta} \bar{m}^\mu.
 \end{aligned} \tag{12}$$

If we know the matrix B , then with (3) we construct the corresponding Lorentz matrix; the inverse problem is to obtain B if we have L , and the answer is [16-18]:

$$B = (B^A_C) = \pm \frac{1}{K} L^\mu_\nu \sigma_\mu^{A\dot{E}} \sigma^\nu_{C\dot{E}}, \quad K \equiv \sqrt{\det(L^\tau_\lambda \sigma_\tau^{Q\dot{F}} \sigma^\lambda_{R\dot{F}})}, \tag{13}$$

in terms of the Infeld-van der Waerden symbols [6, 11, 19-23].

We have that any Lorentz matrix can be generated with the Infeld-van der Waerden symbols via the following expression [9, 23-25]:

$$\begin{aligned}
 L^\mu_\nu &= \sigma^\mu_{A\dot{R}} B^A_C \sigma_\nu^{C\dot{E}} B_E^{\dagger\dot{R}}, \quad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \\
 B^\dagger &= \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}, \quad \alpha \delta - \beta \gamma = 1,
 \end{aligned} \tag{14}$$

then:

$$\begin{aligned}
 L^\mu_\nu &= \sigma^\mu_{A1} B^A_C \sigma_\nu^{C\dot{E}} B_E^{\dagger1} + \sigma^\mu_{A2} B^A_C \sigma_\nu^{C\dot{E}} B_E^{\dagger2} = \\
 &= {}_1P^\mu_C {}_1Q^C_\nu + {}_2P^\mu_C {}_2Q^C_\nu,
 \end{aligned} \tag{15}$$

with the rectangular matrices:

$$({}_rP^\mu_C)_{4 \times 2} = (\sigma^\mu_{A\dot{r}} B^A_C), \quad ({}_rQ^C_\nu)_{2 \times 4} = (\sigma_\nu^{C\dot{E}} B_E^{\dagger\dot{r}}), \quad r = 1, 2. \tag{16}$$

It is simple to construct the matrices (16), in fact

$$({}_1P^\mu{}_c) = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \\ i\gamma & i\delta \\ \alpha & \beta \end{pmatrix}, \quad ({}_1Q^c{}_\nu) = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha^* & \beta^* & i\beta^* & \alpha^* \\ \beta^* & \alpha^* & -i\alpha^* & -\beta^* \end{pmatrix}, \quad (17)$$

$$({}_2P^\mu{}_c) = \frac{1}{\sqrt{2}} \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \\ -i\alpha & -i\beta \\ -\gamma & -\delta \end{pmatrix}, \quad ({}_2Q^c{}_\nu) = \frac{1}{\sqrt{2}} \begin{pmatrix} \gamma^* & \delta^* & i\delta^* & \gamma^* \\ \delta^* & \gamma^* & -i\gamma^* & -\delta^* \end{pmatrix},$$

hence from (15) and (17) the Lorentz matrix is factored in the form:

$$(L^\mu{}_\nu) = ({}_1P \quad {}_2P) \begin{pmatrix} {}_1Q \\ {}_2Q \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \gamma & \delta & \alpha & \beta \\ i\gamma & i\delta & -i\alpha & -i\beta \\ \alpha & \beta & -\gamma & -\delta \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} & i\bar{\beta} & \bar{\alpha} \\ \bar{\beta} & \bar{\alpha} & -i\bar{\alpha} & -\bar{\beta} \\ \bar{\gamma} & \bar{\delta} & i\bar{\delta} & \bar{\gamma} \\ \bar{\delta} & \bar{\gamma} & -i\bar{\gamma} & -\bar{\delta} \end{pmatrix}, \quad (18)$$

in harmony with (4).

Concluding Remarks: With the factorization (4) it is possible to deduce the generating orthogonal matrix of 3-rotations in terms of the Euler angles [26]. The splitting (4) allows to obtain the interesting matrix [27]:

$$S = \begin{pmatrix} A & E \\ E & A \end{pmatrix}, \quad A = \frac{1}{2} \begin{pmatrix} \bar{\alpha} + \delta & \bar{\beta} - \gamma \\ \bar{\gamma} - \beta & \alpha + \bar{\delta} \end{pmatrix}, \quad E = \frac{1}{2} \begin{pmatrix} \bar{\alpha} - \delta & \bar{\beta} + \gamma \\ \beta + \bar{\gamma} & \bar{\delta} - \alpha \end{pmatrix}, \quad (19)$$

which gives the transformation of the Dirac 4-spinor ψ under Lorentz mappings [28-30], that is, $\tilde{\psi} = S \psi$.

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SOME RESULTS ON F-POLYGROUPS

M.M. Zahedi and A. Hasankhani

Department of Mathematics
Kerman University
Kerman, Iran

Abstract

In this note first a product between two F-subpolygroups of a given F-polygroup is defined, and then a necessary and sufficient condition is given in order that the product be an F-subpolygroup . Then by considering the notion of homomorphism of F-polygroups, some related results are obtained. Finally the concept of quotient F-polygroup is defined and then some theorems are proved.

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1. INTRODUCTION

Zadeh in 1965 [9] introduced the notion of fuzzy subsets of a nonempty set A as a function from A to $[0,1]$. Rosenfeld in 1971 [8] defined fuzzy subgroups and obtained some basic results. A polygroup is a completely regular, reversible in itself multigroups in the sense of Dresher and Ore [5]. These system occur naturally in the study of algebraic logic [2,3]. Some algebraic and combinatorial properties were developed in [4] by S. D. Comer. Ioulidis in 1981 [7] studied the concept of polygroup, which is a generalization of the concept of ordinary group. Zahedi, Bolurian and Hasankhani in 1995 [10] introduced the concept of a fuzzy subpolygroup which is a generalization of the concept of a fuzzy subgroup. Zahedi and Hasankhani [11] defined the notion of F-polygroup, which is a generalization of polygroups.

In this note we follow [11]: first we define a product between two F-subpolygroups of a given F-polygroup and then we obtain a necessary and sufficient condition for that product begin an F-subpolygroups. Then we show that the inverse image of a (weak normal) F-subpolygroup under a given homomorphism of F-polygroups is also a (weak normal) F-subpolygroup, and under suitable conditions the homomorphic image of any (weak normal) F-subpolygroup is also a (weak normal) F-subpolygroups. Finally we define the notion of quotient F-polygroup and prove some related results.

2. PRELIMINARIES

Throughout this paper I is the unit interval $[0,1] \subseteq \mathbb{R}$, and if $\mu \in I^A$, then by $\text{supp}(\mu)$ we mean the set $\{x \in A \mid \mu(x) \neq 0\}$. Let $\mu, \eta \in I^A$. Then

$\mu \leq \eta$ iff $\mu(x) \leq \eta(x)$, For all $x \in A$.

(2.1) DEFINITION. Let μ, η and $\mu_\alpha \in I^A$ where α is in the index set Λ . We define the fuzzy subsets $\mu \cap \eta, \mu \cup \eta, \bigcap_{\alpha \in \Lambda} \mu_\alpha$ and $\bigcup_{\alpha \in \Lambda} \mu_\alpha$ as follows:

- (i) $(\mu \cap \eta)(x) = \min(\mu(x), \eta(x))$,
- (ii) $(\mu \cup \eta)(x) = \max(\mu(x), \eta(x))$,
- (iii) $(\bigcap_{\alpha \in \Lambda} \mu_\alpha)(x) = \inf_{\alpha \in \Lambda} \mu_\alpha(x)$,
- (iv) $(\bigcup_{\alpha \in \Lambda} \mu_\alpha)(x) = \sup_{\alpha \in \Lambda} \mu_\alpha(x)$, for all $x \in A$.

(2.2) DEFINITION. Let $f : A \rightarrow B$ be a function and $\mu \in I^A, \eta \in I^B$. Then the functions $f(\mu)$ and $f^{-1}(\eta)$ which belong respectively to I^B and I^A are defined as follows:

- (i) $f(\mu)(b) = \begin{cases} \sup_{a \in f^{-1}(b)} \mu(a) & \text{if } f^{-1}(b) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}, \text{ for all } b \in B,$
- (ii) $f^{-1}(\eta)(a) = \eta(f(a))$, for all $a \in A$.

(2.3) DEFINITION. Let $a \in A, t \in I$. Then by a fuzzy point a_t of A we mean the fuzzy subset of A given below:

$$a_t(x) = \begin{cases} t & \text{if } x = a \\ 0 & \text{otherwise} \end{cases}.$$

(2.4) DEFINITION (See [2,3,4,7]). Let “ $\circ$ ” be a hyperoperation on A . Then $(A, \circ)$ is called a polygroup iff

- (i) $x\circ(y\circ z) = (x\circ y)\circ z \quad \forall x, y, z \in A$,
- (ii) There exists an element $e \in A$ such that $x\circ e = e\circ x = \{x\}, \quad \forall x \in A$.
(e is called the identity element of A .),
- (iii) for each $x \in A$, there exists a unique element $x' \in A$ such that

$e \in xox' \cap x'ox$. (x' is called the inverse of x and is denoted by x^{-1} .),

(iv) $z \in xoy \Rightarrow x \in zoy^{-1} \Rightarrow y \in x^{-1}oz$, $\forall x, y, z \in A$.

For any subset A of X , we let χ_A denote the characteristic function of A .

(2.5) DEFINITION. Let $A \neq \emptyset$ and $I_*^A = I^A \setminus \{o\}$, where o is the function which is identically 0. Then

(i) by an F-hyperoperation “ $*$ ” on A we mean a function from $A \times A$ to I_*^A ; in other words for any $a, b \in A$, $a * b$ is a non-empty fuzzy subset of A .

(ii) if $\mu, \eta \in I_*^A$, then $\mu * \eta \in I_*^A$ is defined by $\mu * \eta = \bigcup_{x \in \text{supp}(\mu), y \in \text{supp}(\eta)} x * y$.

(2.6) NOTATION. Let $\mu \in I_*^A$, $B, C \in P^*(A)$ and $a \in A$. Then

(i) $a * \mu$ and $\mu * a$ denote $\chi_{\{a\}} * \mu$ and $\mu * \chi_{\{a\}}$ respectively,

(ii) $a * B, B * a, \mu * B, B * \mu$ and $B * C$ denote $\chi_{\{a\}} * \chi_B, \chi_B * \chi_{\{a\}}, \mu * \chi_B, \chi_B * \mu$ and $\chi_B * \chi_C$ respectively.

(2.7) DEFINITION. Let $\mathcal{F}$ be a non-empty set and “ $*$ ” an F-hyperoperation on $\mathcal{F}$. Then $(\mathcal{F}, *)$ is called an F-polygroup iff

(i) $(x * y) * z = x * (y * z)$, $\forall x, y, z \in \mathcal{F}$,

(ii) there exists an element $e_F \in \mathcal{F}$ such that $x \in \text{supp}(x * e_F \cap e_F * x)$, $\forall x \in \mathcal{F}$ (In this case we say e_F is an F-identity element of $\mathcal{F}$.),

(iii) for each $x \in \mathcal{F}$, there exists a unique element $x' \in \mathcal{F}$ such that $e_F \in \text{supp}(x * x' \cap x' * x)$. (x' is called the F-inverse of x and is denoted by x_F^{-1} .),

(iv) $z \in \text{supp}(x * y) \Rightarrow x \in \text{supp}(z * y_F^{-1}) \Rightarrow y \in \text{supp}(x_F^{-1} * z)$, $\forall x, y, z \in \mathcal{F}$. (This property is called the F -reversibility of $\mathcal{F}$ with respect to $*$.)

When there is no ambiguity, for simplicity of notation we use e and x^{-1} instead of e_F and x_F^{-1} respectively.

If $\mathcal{F}$ is an F -polygroup and $x * y = y * x$, $\forall x, y \in \mathcal{F}$, then $\mathcal{F}$ is said to be Abelian.

Henceforth $\mathcal{F}$ will denote an F -polygroup with hyperoperation “ $*$ ” and e will denote the F -identity of $\mathcal{F}$.

REMARK. If $e \in \text{supp}(x * y)$, then y is called a right F -inverse of x and x is called a left F -inverse of y with respect to “ $*$ ” in $\mathcal{F}$.

(2.8) THEOREM[11]. Let $(A, *)$ be an F -polygroup and $\forall x \in A$, $x * e = e * x = \chi_{\{x\}}$. Then $(A, \odot)$ is a polygroup where $x \odot y = \text{supp}(x * y)$, $\forall x, y \in A$. ($\odot$ is called the hyperoperation extracted from $*$.)

(2.9) DEFINITION[11]. If $(x * e)(x) = (e * x)(x) = 1$, $\forall x \in \mathcal{F}$, then we say e is of degree 1.

(2.10) THEOREM[11]. Let $(\mathcal{F}, *)$ be an F -polygroup. Then

- (i) $e^{-1} = e$ and e is unique. Also $\text{supp}(e * e) = \{e\}$,
- (ii) $(x^{-1})^{-1} = x$, $\forall x \in \mathcal{F}$,
- (iii) $\bigcup_{x \in \text{supp}(\mu_1)} x * \mu_2 = \mu_1 * \mu_2 = \bigcup_{y \in \text{supp}(\mu_2)} \mu_1 * y$, $\forall \mu_1, \mu_2 \in I_*^{\mathcal{F}}$,
- (iv) $(\mu_1 * \mu_2) * \mu_3 = \mu_1 * (\mu_2 * \mu_3)$, $\forall \mu_1, \mu_2, \mu_3 \in I_*^{\mathcal{F}}$.

(2.11) DEFINITION[11]. Let $\mathcal{F}_1, \mathcal{F}_2$ be two F -polygroups and $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be a function such that $f(e_1) = e_2$. Then

- (i) f is called a homomorphism iff $f(x * y) \leq f(x) * f(y)$, $\forall x, y \in \mathcal{F}_1$
- (ii) f is called a strong homomorphism iff $f(x * y) = f(x) * f(y)$, $\forall x, y \in \mathcal{F}_1$

(2.12) DEFINITION[11]. Let $\emptyset \neq H \subseteq \mathcal{F}$. Then H is called an F -subpolygroup iff

- (i) if $x \in H$, then $x^{-1} \in H$
- (ii) $\text{supp}(x * y) \subseteq H$, $\forall x, y \in H$.

In this case we write: $H <_{F-P} \mathcal{F}$.

Note that condition (ii) of above definition is equivalent to $x * y \leq \chi_H$, $\forall x, y \in H$.

(2.13) LEMMA[11]. Let $\emptyset \neq H \subseteq \mathcal{F}$. Then $H <_{F-P} \mathcal{F}$ if and only if $\text{supp}(x * y^{-1}) \subseteq H$, $\forall x, y \in H$.

(2.14) DEFINITION[11] Let $H <_{F-P} \mathcal{F}$. Then

(i) H is said to be weak normal in $\mathcal{F}(H \triangleleft_{F-P}^w \mathcal{F})$ iff

$$x * H * x^{-1} \leq \chi_H, \quad \forall x \in \mathcal{F}, \quad (1)$$

(ii) H is said to be normal in $\mathcal{F}(H \triangleleft_{F-P} \mathcal{F})$ iff $x * H * x^{-1} = \chi_H$, $\forall x \in \mathcal{F}$.

Note that (1) is equivalent to:

$$\text{supp}(x * H * x^{-1}) \subseteq H, \quad \forall x \in \mathcal{F}.$$

(2.15) THEOREM[11]. Let $e \in \mathcal{F}$ be of degree 1. Then

$$N \triangleleft_{F-P}^w \mathcal{F} \text{ iff } N \triangleleft_{F-P} \mathcal{F}.$$

3. ON F-SUBPOLYGROUPS

(3.1) LEMMA. Let $n \in N$ and $w, a_1, \dots, a_n \in \mathcal{F}$. If $w \in \text{supp}(a_1 * \dots * a_n)$, then $w^{-1} \in \text{supp}(a_n^{-1} * \dots * a_1^{-1})$.

(3.2) THEOREM. Let $H, K <_{F-P} \mathcal{F}$ and $H \odot K = \bigcup_{x \in H, y \in K} \text{supp}(x * y)$. Then

- (i) $H \odot K <_{F-P} \mathcal{F}$ if and only if $H \odot K = K \odot H$,
- (ii) if $K \triangleleft_{F-P}^w \mathcal{F}$, then $H \odot K <_{F-P} \mathcal{F}$,
- (iii) if $H, K \triangleleft_{F-P}^w \mathcal{F}$, then $H \odot K \triangleleft_{F-P}^w \mathcal{F}$.

Proof. (i) Let $H \odot K <_{F-P} \mathcal{F}$ and $w \in H \odot K$ be an arbitrary element. Then $w^{-1} \in H \odot K$. Therefore there is $h \in H$ and $k \in K$ such that $w^{-1} \in \text{supp}(h * k)$. Thus by Lemma 3.1 we have $w \in \text{supp}(k^{-1} * h^{-1})$. Consequently $w \in K \odot H$, and hence $H \odot K \subseteq K \odot H$. Similarly $K \odot H \subseteq H \odot K$. That is $H \odot K = K \odot H$.

Conversely, suppose $H \odot K = K \odot H$. Since $e \in \text{supp}(e * e) \subseteq H \odot K$, hence $H \odot K \neq \emptyset$. Let $a, b \in H \odot K$. Then there exist $k, k' \in K$, $h, h' \in H$ such that

$$a \in \text{supp}(h * k) \text{ and } b \in \text{supp}(h' * k').$$

So

$$\begin{aligned} a * b^{-1} &\leq h * k * k'^{-1} * h'^{-1}, \quad \text{since } b^{-1} \in \text{supp}(k'^{-1} * h'^{-1}) \\ &\leq h * K * h'^{-1}, \quad \text{since } K <_{F-P} \mathcal{F} \\ &= \bigcup_{t \in \text{supp}(K * h'^{-1})} (h * t). \end{aligned} \tag{2}$$

Now let $s \in \text{supp}(a * b^{-1})$. Then there exists $t' \in \text{supp}(K * h'^{-1})$ such that $s \in \text{supp}(h * t')$. Therefore $t' \in \text{supp}(k * h'^{-1})$ for some $k \in K$. So $(h * t')(s) \leq h * (k * h'^{-1})(s)$, and there is $r \in \text{supp}(k * h'^{-1})$ such the $s \in \text{supp}(h * r)$. Consequently $r \in K \odot H = H \odot K$. Thus $r \in \text{supp}(h_2 * k_2)$, for some $h_2 \in H$ and $k_2 \in K$. Hence

$$\begin{aligned} 0 < (h * r)(s) &\leq (h * h_2 * k_2)(s), \\ &\leq (H * k_2)(s), \quad \text{since } H <_{F-P} \mathcal{F}. \end{aligned}$$

Thus $s \in H \odot K$, which shows that $H \odot K <_{F-P} \mathcal{F}$, by Lemma 2.13.

(ii) Let $s \in H \odot K$, then there are $h \in H$ and $k \in K$ such that $s \in \text{supp}(h * k)$. Thus $s \in \text{supp}(h * k * h^{-1} * h)$. Since $K \triangleleft_{F-P}^w \mathcal{F}$, we get

$s \in \text{supp}(K * h) \subseteq K \odot H$. So $H \odot K \subseteq K \odot H$, and similarly $K \odot H \subseteq H \odot K$. Hence $H \odot K = K \odot H$ which implies that $H \odot K <_{F-P} \mathcal{F}$, by (i).

(iii) Let $x \in \mathcal{F}$ and $a \in \text{supp}(x * (H \odot K) * x^{-1})$. Thus we have $a \in \text{supp}(x * t * x^{-1})$, for some $t \in H \odot K$. Hence $t \in \text{supp}(h * k)$, for some $h \in H$ and $k \in K$. So $a \in \text{supp}(x * h * k * x^{-1})$. Since $H, K \triangleleft_{F-P}^w \mathcal{F}$ we get $a \in \text{supp}(x * h * x^{-1} * x * k * x^{-1}) \subseteq H \odot K$. Consequently $H \odot K \triangleleft_{F-P}^w \mathcal{F}$.

(3.3) COROLLARY. Let $e \in \mathcal{F}$ be of degree 1 and $H, K \triangleleft_{F-P}^w \mathcal{F}$. Then $H \odot K \triangleleft_{F-P} \mathcal{F}$.

Proof. The proof easily follows from Theorem 2.15 and 3.2(iii).

(3.4) QUESTION. If $H, K \triangleleft_{F-P} \mathcal{F}$, is $H \odot K \triangleleft_{F-P} \mathcal{F}$?

(3.5) DEFINITION. Let $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be a strong homomorphism of F -polygroups. Then f is called a zero-invariant iff

$$(x * y)(z) = 0 \implies f(x * y)(f(z)) = 0, \quad \forall x, y, z \in \mathcal{F}_1.$$

(3.6) THEOREM. Let $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be a homomorphism of F -polygroups. Then

(i) if $K <_{F-P} \mathcal{F}_2 (K \triangleleft_{F-P}^w \mathcal{F}_2)$, then $f^{-1}(K) <_{F-P} \mathcal{F}_1, (f^{-1}(K) \triangleleft_{F-P}^w \mathcal{F}_1)$,

(ii) if f is strong (onto and zero-invariant) and $H <_{F-P} \mathcal{F}_1 (H \triangleleft_{F-P}^w \mathcal{F}_1)$, then $f(H) <_{F-P} \mathcal{F}_2 (f(H) \triangleleft_{F-P}^w \mathcal{F}_2)$.

(3.7) REMARK. Consider a group G with order greater than 1 and such that $x^2 = e, \quad \forall x \in G$. Then as is shown in Examples 3.5 and 4.9 of [11], there are two F -polygroups structure on G as follow:

(i) $(x * y)(z) = e_\alpha(xyz), \quad \forall x, y, z \in G,$

(ii) $(x *' y)(z) = \begin{cases} e_1(xyz) & \text{if } z = e \\ e_\alpha(xyz) & \text{if } z \neq e, \end{cases}$

where α is a fixed element in $(0, 1)$, and e_α, e_1 are fuzzy points in G .

Now let $f : (G, *) \rightarrow (G, *')$ be the identity map. Then $f^{-1}(\{e\}) \ntriangleleft_{F-P}(G, *)$, while $\{e\} \triangleleft_{F-P}(G, *')$. (see Example 4.9 of [11].)

(3.8) QUESTION. Is the following statement true?

Let $\mathcal{F}_1, \mathcal{F}_2$ be two arbitrary F-polygroups. If $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is a zero-invariant and onto, then the homomorphic image of a normal F-subpolygroup is also normal F-subpolygroup.

4. QUOTIENT F-POLYGROUP.

(4.1) LEMMA. Let $N \triangleleft_{F-P} \mathcal{F}$. Then

- (i) $N * x = x * N, \quad \forall x \in \mathcal{F},$
- (ii) $x * N = \chi_N$ iff $x \in N,$
- (iii) $N * N = \chi_N.$

NOTATION: Let $H \triangleleft_{F-P} \mathcal{F}$. Then $\mathcal{F}/H$ denotes the set of all $x * H$, where $x \in \mathcal{F}$, i.e. $\mathcal{F}/H = \{x * H : x \in \mathcal{F}\}.$

(4.2.) THEOREM. Let $H \triangleleft_{F-P} \mathcal{F}$. Define the F -hyperoperation " $\square$ " on $\mathcal{F}/H$ as follows:

$$\begin{aligned} \square : \mathcal{F}/H \times \mathcal{F}/H &\longrightarrow I_*^{\mathcal{F}/H} \\ (x * H, y * H) &\longmapsto x * H \square y * H \end{aligned}$$

where

$$(x * H \square y * H)(z * H) = (x * y * H)(z), \quad \forall z * H \in \mathcal{F}/H.$$

Then $(\mathcal{F}/H, \square)$ is an F -polygroup called the quotient F -polgroup.

(4.3) REMARK. The proof of uniqueness of $x^{-1} * H$ in the Theorem 4.2 shows that the right F -inverse of any element $x * H$ of $\mathcal{F}/H$ is unique and similarly, by imposing a suitable modification, the left F -inverse of any element $x * H$ of $\mathcal{F}/H$ is also unique.

(4.4) THEOREM. Let $H \triangleleft_{F-P}^w \mathcal{F}$ and the identity of $\mathcal{F}$ be of degree 1. Then $(\mathcal{F}/H, \odot)$ is a polygroup, where “ $\odot$ ” is the hyperoperation extracted from “ $\square$ ” (see Theorem 2.8).

Proof. Let $x * H \in \mathcal{F}/H$ be arbitrary. We prove that

$$x * H \square e * H = e * H \square x * H = \chi_{\{x * H\}}.$$

We have, $(x * H \square e * H)(x * H) = (x * e * H)(x) = (x * H)(x) =$, since $(x * e)(x) = 1$.

Now let $y * H \in \text{supp}(x * H \square e * H)$. Then by the reversibility of $H/\mathcal{F}$ with respect to “ $\square$ ” we get $e * H \in \text{supp}(x^{-1} * H \square y * H)$. Therefore Remark 4.3 implies that $x * H = y * H$. Hence

$x * H \square e * H = \chi_{\{x * H\}}$. Similarly $e * H \square x * H = \chi_{\{x * H\}}$. The proof of theorem follows, by Theorem 2.8.

(4.5) LEMMA. Let $\{H_i\}_{i \in \Lambda}$ be a family of F -subpolygroups of $\mathcal{F}$. Then $H = \bigcap_{i \in \Lambda} H_i$ is an F -subpolygroups of $\mathcal{F}$.

Proof. The proof easily follows from Lemma 2.13.

(4.6) DEFINITION. Let $\emptyset \neq S \subseteq \mathcal{F}$. Then by the F -subpolygroup $\langle S \rangle$ generated by S , we mean the intersection of all F -subpolygroups of $\mathcal{F}$, containing S .

(4.7) THEOREM. Let $e \in S \subseteq \mathcal{F}$. Then $\langle S \rangle = \bigcup_{n \in \mathbb{N}, a_i \in S \cup S^{-1}} \text{supp}(a_1 * \dots * a_n)$

$\cdots * a_n)$, where $S^{-1} = \{x^{-1} : x \in S\}$.

Proof. The proof follows from Lemma 3.1.

(4.8) THEOREM. Let $C = \{z \in \mathcal{F} : z \in \text{supp}(x*y*x^{-1}*y^{-1}), \text{ for some } x, y \in \mathcal{F}\}$. Then $\mathcal{F}' = \langle C \rangle$ is a weak normal F -subpolygroup of $\mathcal{F}$ called the commutator F -subpolygroup of $\mathcal{F}$. Moreover if $W \triangleleft_{F-P} \mathcal{F}$ and $\mathcal{F}' \subseteq W$, then $(\mathcal{F}/W, \square)$ is an Abelian F -polygroup. In particular if the identity of $\mathcal{F}$ is of degree 1, then $\mathcal{F}/\mathcal{F}'$ is Abelian.

Proof. By Lemma 4.5, $\mathcal{F}' <_{F-P} \mathcal{F}$. Now we show that $\mathcal{F}' \triangleleft_{F-P}^w \mathcal{F}$. To this end let $y \in \mathcal{F}'$ and $x \in \mathcal{F}$ be arbitrary. We have

$$\begin{aligned} x * y * x^{-1} &\leq x * y * x^{-1} * e \leq x * y * x^{-1} * y^{-1} * y \\ &= \bigcup_{t \in \text{supp}(x*y*x^{-1}*y^{-1})} t * y \\ &\leq \chi_{\mathcal{F}'}, \quad \text{since } \mathcal{F}' <_{F-P} \mathcal{F}. \end{aligned}$$

Therefore $\mathcal{F}' \triangleleft_{F-P}^w \mathcal{F}$.

Now let $W \triangleleft_{F-P} \mathcal{F}$ and $\mathcal{F}' \subseteq W$. We will prove that $\mathcal{F}/W$ is Abelian. By hypothesis, $\chi_{\mathcal{F}'} \leq \chi_W$, which implies that $y^{-1} * x^{-1} * y * x * W \leq \mathcal{F}' * W \leq W * W = \chi_W$, $\forall x, y \in \mathcal{F}$. Thus

$$\begin{aligned} y * x * W &\leq x * x^{-1} * y * x * W \\ &\leq x * y * y^{-1} * x^{-1} * y * x * W \\ &\leq x * y * W, \quad \forall x, y \in \mathcal{F}. \end{aligned}$$

Similarly $x * y * W \leq y * x * W$, for all $x, y \in \mathcal{F}$. Hence

$$\forall x, y \in \mathcal{F}, \quad x * y * W = y * x * W,$$

which implies that $\mathcal{F}/W$ is Abelian.

(4.9) COROLLARY. Let the identity of $\mathcal{F}$ be of degree 1, $W \triangleleft_{F-P}^w \mathcal{F}$ and $\mathcal{F}' \subseteq W$. Then $(\mathcal{F}/W, \odot)$ is an Abelian polygroup, where “ $\odot$ ” is the hyperoperation extracted from “ $\square$ ”

Proof. The proof follows from Theorem 4.4 and 4.8 and definition of “ $\odot$ ”.

(4.10) THEOREM. Let for all $x \in \mathcal{F}$, $\text{supp}(x * x^{-1}) = \{e\}$, $W \triangleleft_{F-P} \mathcal{F}$ and $(\mathcal{F}/W, \square)$ be an Abelian F -polygroup. Then $\mathcal{F}' \subseteq W$.

Proof. It is enough to show that $C \subseteq W$, where C is defined in Theorem 4.8. To this end let $y \in C$ be arbitrary. Then there exist $a, b \in \mathcal{F}$ such that $y \in \text{supp}(a * b * a^{-1} * b^{-1})$. Since $e \in W$, we have $y \in \text{supp}(a * b * a^{-1} * b^{-1} * W)$. Hence $y \in W$.

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QUOTIENTS OF $P-H_V$ -RINGS

Stefanos Spartalis

Department of Mathematics, School of Engineering
Democritus University of Thrace
67100 Xanthi, Greece

Abstract

The object of this paper is the H_V -ring. There are generalizations of the known hyperrings where axioms are replaced by the weak ones. That is, instead of the equality on sets, one has non-empty intersections. A wide class of H_V -rings is the class of the $P-H_V$ -rings. The isomorphism theorems are proved. The fundamental relation, i.e. the smallest equivalence relation for which the quotient is a ring, is investigated.

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1. INTRODUCTION

An algebraic hyperstructure is a pair $\Delta = (H, \mathcal{E})$ where H is a non empty set and $\mathcal{E}$ is a finite set of hyperoperations, that is

if $"\cdot" \in \mathcal{E}$ then, $\cdot : H \times H \rightarrow \mathcal{P}(H) \setminus \emptyset : (x, y) \rightarrow xy$.

A hyperstructure is called a H_V -structure [9] if it satisfies the weak axioms. For example, the weak associativity, for all x, y, z in H , is

$$x(yz) \cap (xy)z \neq \emptyset,$$

and the left and the right weak distributivity (for the hyperoperations $\cdot, +$) is

$$x(y+z) \cap (xy+xz) \neq \emptyset, \quad (x+y)z \cap (xz+yz) \neq \emptyset.$$

A H_V -ring [7],[6] is the hyperstructure $\langle R, +, \cdot \rangle$ where $\langle R, + \rangle$ is a H_V -group (i.e. the addition is weak associative and reproductive $x+H = H = H+x$, for all $x \in H$), $\langle R, \cdot \rangle$ is a H_V -semigroup and $(\cdot)$ is weak distributive with respect to $(+)$.

If only the $(\cdot)$ is a hyperoperation, then the H_V -ring R is called multiplicative and it is called additive, if $(\cdot)$ is single-valued.

Left (respectively, right) H_V -ideal in a H_V -ring $(R, +, \cdot)$ is called a subset $J \subset R$, where $\langle J, + \rangle$ is a H_V -group and $JR \subset J$ (respectively, $RJ \subset J$).

The fundamental relations β^* and γ^* are defined on H_V -rings in a similar way as in hyperrings [1],[7]. The relation γ^* is the smallest equivalence relation such that the quotient R/γ^* is a ring. Then γ^* is the transitive closure of the relation γ [6] defined as follows

$$x\gamma y \quad \text{if and only if} \quad \{x, y\} \in a, \quad a \in A,$$

where A is the set of all finite polynomials of elements of R .

A H_V -ring R is called a H_V -field, if its fundamental ring R/γ^* is a field.

A wide class of H_V -rings is the class of H_V -rings with P -hyperoperations or P - H_V -rings introduced in [6]. In this paper we investigate the quotients of P - H_V -rings, with respect to H_V -ideals and to the fundamental relation γ^* . The isomorphism theorems are proved.

2. QUOTIENTS WITH RESPECT TO H_V -IDEALS

Let $(H, \cdot)$ be a semigroup and $P \subset H$, $P \neq \emptyset$. The P -hyperoperations [7] are defined, for all $x, y \in H$, as follows:

$$xP_C^*y = xPy, \quad xP_R^*y = xyP, \quad xP_L^*y = Pxy.$$

Recall that $\langle H, P_C^* \rangle$ is a semihypergroup and it is a hypergroup if and only if $(H, \cdot)$ is a group [10].

PROPOSITION 2.1

Let $(H, \cdot)$ be a semigroup and $P \subset H$. If, for all $x \in H$, $xPP \cap PxP \neq \emptyset$, then $\langle H, P_R^* \rangle$ is a H_V -semigroup.

Proof

For all x, y, z in H we have

$$xP_R^*(yP_R^*z) = xP_R^*(yzP) = xyzPP,$$

$$(xP_R^*y)P_R^*z = (xyP)P_R^*z = xyPzP.$$

But, since $zPP \cap PzP \neq \emptyset$, it follows that

$$xP_R^*(yP_R^*z) \cap (xP_R^*y)P_R^*z \neq \emptyset.$$

REMARK 2.1.1

Denote $Z(H)$ the center of $(H, \cdot)$. Notice that in [6] (Proposition 3.1) the condition $Z(H) \cap P \neq \emptyset$ stands instead of $xPP \cap PxP \neq \emptyset$, for all $x \in R$. Obviously, Proposition 2.1 is more general of the one in [6]. Moreover, if $P \subset Z(H)$, then $P_C^* = P_R^* = P_L^*$.

COROLLARY 2.2

Let $(H, \cdot)$ be a monoid with unit 1. $\langle H, P_R^* \rangle$ is a H_V -semigroup if and only if $xPP \cap PxP \neq \emptyset$, for all $x \in H$.

It is obtained that the H_V -semigroup $\langle H, P_R^* \rangle$ is a H_V -group if and only if $(H, \cdot)$ is a group (see also [10]). Moreover, the subset B is a H_V -subgroup if $\langle B, P_R^* \rangle$ is a H_V -group and it is a P - H_V -subgroup if $P \subset B$ [8].

LEMMA 2.3

Let 1 be the unit of the group $(H, \cdot)$ and $1 \in P \subset H$. The subset B of the H_V -group $\langle H, P_R^* \rangle$ is a P - H_V -subgroup if and only if $P \subset B$ and B is a subgroup of $(H, \cdot)$.

Proof

Analogous to the Lemma 6.1 of [3].

REMARK

Similar results hold for the hyperoperation P_L^* . For example, the sufficient condition for the weak associativity is $PPx \cap P_xP \neq \emptyset$, for all $x \in R$.

Now consider any ring $(R, +, \cdot)$ and take $P_1, P_2 \subset R$, $P_1 \neq \emptyset \neq P_2$. We shall make use of the following right P -hyperoperations:

$$xP_1^*y = x+y+P_1, \quad xP_2^*y = xyP_2, \quad \text{for all } x, y \in R.$$

Remark that, $P_{1C}^* \equiv P_{1R}^* \equiv P_{1L}^*$ and this hyperoperation is of constant length [2]. On the other hand using the multiplication we can define three different hyperoperations xP_2^*y , xyP_2 and P_2xy . For results using xP_2^*y see [3], [4]. The results for P_2xy are similar.

THEOREM 2.4

Let $(R, +, \cdot)$ be a ring and $P_1, P_2 \subset R$. If $0 \in P_1$ and for all $x \in R$, $xP_2^*P_2 \cap P_2xP_2 \neq \emptyset$, then $\langle R, P_1^*, P_2^* \rangle$ is a H_V -ring called P - H_V -ring.

Proof

$\langle R, P_1^* \rangle$ is a P -hypergroup [8] and $\langle R, P_2^* \rangle$ is a H_V -semigroup (Proposition 2.1). Moreover, since the necessary and sufficient condition for the weak distributivity is $0 \in P_1$ [6] it follows that $\langle R, P_1^*, P_2^* \rangle$ is a H_V -ring.

Let J be a two-sided H_V -ideal of $\langle R, P_1^*, P_2^* \rangle$. Since,

$$0 \in RP_2^*J \cap JP_2^*R \subset J \quad \text{and} \quad P_1 = OP_1^*0 \subset J,$$

we have that $\langle J, P_1^* \rangle$ is a $P-H_V$ -subgroup of $\langle R, P_1^* \rangle$. Therefore, by using Lemma 2.3, it follows that J is a subgroup of $(R, +)$ and so, for all $x \in R$, $JP_1^*x = J+x = xP_1^*J$. Moreover, the addition $(\oplus)$ and the multiplication $(\odot)$ between classes are defined, for all $x, y \in R$, in a usual manner:

$$(JP_1^*x) \oplus (JP_1^*y) = \{JP_1^*z : z \in (JP_1^*x)P_1^*(JP_1^*y)\} = \{J+x+y\},$$

$$(JP_1^*x) \odot (JP_1^*y) = \{J+w : w \in (JP_1^*x)P_2^*(JP_1^*y)\} = \{J+w : w \in xyP_2\}.$$

THEOREM 2.5

If $\langle R, P_1^*, P_2^* \rangle$ is a $P-H_V$ -ring and J is a two-sided H_V -ideal, then $\langle R/J, \oplus, \odot \rangle$ is a multiplicative H_V -ring.

Proof

Obviously $\langle R/J, \oplus \rangle$ is an abelian group. Moreover, $\langle R/J, \odot \rangle$ is a H_V -semigroup because $(\odot)$ is well defined and for all $x, y, z \in R$ we have

$$(JP_1^*x) \odot [(JP_1^*y) \odot (JP_1^*z)] = \{J+v : v \in xyzP_2P_2\},$$

$$[(JP_1^*x) \odot (JP_1^*y)] \odot (JP_1^*z) = \{J+u : u \in xyP_2zP_2\}.$$

But, since $zP_2P_2 \cap P_2zP_2 \neq \emptyset$, it follows that the multiplication is weak associative.

Finally, $(JP_1^*x) \odot [(JP_1^*y) \oplus (JP_1^*z)] = \{J+v : v \in x(y+z)P_2\} \subset$

$$\subset \{J+u : u \in xyP_2 + xzP_2\} = [(JP_1^*x) \odot (JP_1^*y)] \oplus [(JP_1^*x) \odot (JP_1^*z)].$$

In the same way the right distributivity is proved and so $\langle R/J, \oplus, \odot \rangle$ is a multiplicative H_V -ring.

THEOREM 2.6

Let $\langle R, P_1^*, P_2^* \rangle$ be a $P-H_V$ -ring over the ring $(R, +, \cdot)$ and J a two-sided ideal of the ring R , containing P_1 . Then $\langle R/J, \oplus, \odot \rangle$ is a multiplicative H_V -ring, which is a $P-H_V$ -ring.

Proof

From hypothesis and Lemma 2.3 we have that J is a two-sided H_V -ideal of $\langle R, P_1^*, P_2^* \rangle$. So, according to the previous theorem, $\langle R/J, \oplus, \odot \rangle$ is a multiplicative H_V -ring.

On the other hand consider the quotient ring $(R/J, +, \cdot)$ and take $L_1 = \{J\}$, $L_2 = \{J+a : a \in P_2\}$. Then, for all $x \in R$, we have

$$(J+x)L_2L_2 = \{J+v : v \in xP_2P_2\}, \quad L_2(J+x)L_2 = \{J+w : w \in P_2xP_2\}.$$

Since, for all $x \in R$, $xP_2P_2 \cap P_2xP_2 \neq \emptyset$, it is implied that $(J+x)L_2L_2 \cap L_2(J+x)L_2 \neq \emptyset$. Consequently, since J is the zero element of R/J , Theorem 2.4 follows that, $\langle R/J, L_1^*, L_2^* \rangle$ is a P - H_V -ring over the ring $(R/J, +, \cdot)$.

Therefore, the hyperoperation $(\odot)$ is the P -hyperoperation L_2^* and $(\oplus)$ is the degenerate P -hyperoperation L_1^* .

THEOREM 2.7

Let $\langle R, P_1^*, P_2^* \rangle$ be a P - H_V -ring and J is a two-sided H_V -ideal of R . If H is a H_V -subring of R containing P_1 , then

$$HP_1^*J/J \cong H/H \cap J.$$

Proof

From Lemma 2.3 it follows that H, J are subgroups of $(R, +)$ and so, $HP_1^*J = H+J$, $H \cap J$ are two groups containing P_1 . Since, $(HP_1^*J)P_2^*(HP_1^*J) = (H+J)(H+J)P_2 \subset HHP_2+J \subset HP_1^*J$ we have that $\langle HP_1^*J, P_1^*, P_2^* \rangle$ is a H_V -subring of $\langle R, P_1^*, P_2^* \rangle$. Moreover, J , $J \cap H$ are respectively, H_V -ideals of the H_V -rings HP_1^*J and H . On the other hand, it is obtained that the quotients

$$HP_1^*J/J = \{J+x : x \in H\} \quad \text{and} \quad H/H \cap J = \{(H \cap J)+y : y \in H\}$$

are multiplicative H_V -rings. We consider the bijection map

$$f : HP_1^*J/J \rightarrow H/H \cap J : J+x \rightarrow (H \cap J)+x$$

The map f is a homomorphism since, for all $x, y \in R$, we have

$$f(J+x \oplus J+y) = f(J+x+y) = (H \cap J)+x+y = f(J+x) \oplus f(J+y),$$

$$\begin{aligned} f[(J+x) \circ (J+y)] &= f(\{J+s : s \in xyP_2\}) = \{(H \cap J)+s : s \in xyP_2\} = \\ &= f(J+x) \circ f(J+y) \end{aligned}$$

Consequently, f is an isomorphism.

THEOREM 2.8

Let J, K be H_V -ideals (two-sided) of the $P-H_V$ -ring $\langle R, P_1^*, P_2^* \rangle$. If $J \subset K$, then

$$(R/J)/(K/J) \cong R/K.$$

Proof

However the quotients R/J , K/J , R/K are multiplicative H_V -rings. Moreover, K/J is a two-sided H_V -ideal of $\langle R/J, \oplus, \odot \rangle$. Indeed, $K/J \subset R/J$ and, for all $x \in K$, $y \in R$, we have

$$(J+x) \odot (J+y) = \{J+z : z \in xP_2^*y\} \subset K/J, \text{ since } xP_2^*y \subset K.$$

Therefore, $K/J \odot R/J \subset K/J$. Similarly, $R/J \odot K/J \subset K/J$. On the other hand it is obtained that $\langle (R/J)/(K/J), \blacksquare, \square \rangle$, where $\blacksquare$ and $\square$ are the usual addition and multiplication of classes, is a multiplicative H_V -ring.

Now consider the map $f : (R/J)/(K/J) \rightarrow R/K : K/J \oplus (J+x) \rightarrow K+x$

Since, for all $x, y \in R$,

$$\begin{aligned} (K/J) \oplus (J+x) &= (K/J) \oplus (J+y) \Leftrightarrow \{J+z : z \in K+x\} = \{J+w : w \in K+y\} \Leftrightarrow \\ &\Leftrightarrow y-x \in K \Leftrightarrow K+x = K+y \end{aligned}$$

it follows that f is well defined and one to one. Obviously, it is onto, so, it remains to prove that f is an isomorphism. Indeed, for all $x, y \in R$ we have

$$\begin{aligned} &f[(K/J \oplus (J+x)) \blacksquare (K/J \oplus (J+y))] = \\ &= f[\{K/J \oplus (J+z) : J+z \in K/J \oplus (J+x) \odot K/J \oplus (J+y)\}] = f(K/J \oplus (J+x+y)) = \\ &= K+x+y = (K+x) \odot (K+y) = f(K/J \oplus (J+x)) \odot f(K/J \oplus (J+y)), \\ &f[(K/J \oplus (J+x)) \square (K/J \oplus (J+y))] = \\ &= f[\{K/J \oplus (J+w) : J+w \in (J+x) \odot (J+y)\}] = \\ &= f[\{K/J \oplus (J+w) : w \in xyP_2\}] = \{K+w : w \in xP_2^*y\} = (K+x) \odot (K+y) = \\ &= f(K/J \oplus (J+x)) \odot f(K/J \oplus (J+y)) \end{aligned}$$

3. THE FUNDAMENTAL RELATION

This paragraph focuses on the quotients with respect to the fundamental relation γ^* . Consider a $P\text{-}H_v\text{-ring}$ $\langle R, P_1^*, P_2^* \rangle$ over the ring $(R, +, \cdot)$ such that $RP_2 \subset P_2$. Denote by A the set of all finite polynomials of elements of R . Then for every $a_i \in A$, $i \in \mathbb{N}$, there exists $r_i \in R$, I_i finite set of indices, $P_{2j} \in \mathcal{P}(P_2)$, $j \in I_i$ and $s_i \in \mathbb{N}$, such that

$$a_i = r_i + \sum_{j \in I_i} P_{2j} + s_i P_1 \quad (3.1)$$

where $s_i P_1 = P_1 + \dots + P_1$, that is s_i -times P_1 .

THEOREM 3.2

Let $\langle R, P_1^*, P_2^* \rangle$ be a $P\text{-}H_v\text{-ring}$ over $(R, +, \cdot)$. If $RP_2 \subset P_2$ and $\langle P_1, P_2 \rangle$ is the subgroup of $(R, +)$ generated by $P_1 \cup P_2$, then for all $x \in R$,

$$\gamma^*(x) \subset \langle P_1, P_2 \rangle + x.$$

Proof

Let $x \in R$ and $y \in \gamma^*(x)$. Then, there exists $z_1, \dots, z_{m+1} \in R$, with $z_1 = y$, $z_{m+1} = x$ and $a_1, \dots, a_m \in A$ such that

$$\{z_i, z_{i+1}\} \subset a_i, \quad i=1, 2, \dots, m$$

Applying (3.1), then for $i=m$ we have

$$x = z_{m+1} \in r_m + \sum_{j \in I_m} P_{2j} + s_m P_1, \quad \text{so,} \quad r_m \in x + t_m (-P_2) + s_m (-P_1),$$

where $t_m = \text{card } I_m$.

Moreover, for all $i = 1, 2, \dots, m$, we have

$$z_{i+1} \in (r_i + \sum_{j \in I_i} P_{2j} + s_i P_1) \cap (r_{i+1} + \sum_{j \in I_{i+1}} P_{2j} + s_{i+1} P_1)$$

and hence, $r_i \in (r_{i+1} + t_{i+1} P_2 + t_i (-P_2) + s_{i+1} P_1 + s_i (-P_1))$,

where $t_{i+1} = \text{card } I_{i+1}$, $t_i = \text{card } I_i$.

From this relation, we obtain

$$y = z_1 \in x + t(P_2 - P_1) + s(P_1 - P_1)$$

where $t = t_1 + \dots + t_m$, $t_i = \text{card} I_i$, $i \in \{1, \dots, m\}$, $s = s_1 + \dots + s_m$.

That means that $y \in x + \langle P_2 \rangle + \langle P_1 \rangle$, hence, $\gamma^*(x) \subset \langle P_1, P_2 \rangle + x$.

THEOREM 3.3

Let $\langle R, P_1^*, P_2^* \rangle$ be a P - H_V -ring over the unitary ring $(R, +, \cdot)$. If P_2 is a right ideal and $\langle P_1 \rangle$ is the subgroup of $(R, +)$ generated by P_1 , then

$$R/\gamma^* = R/\langle P_1 \rangle + P_2.$$

Proof

Let $x \in R$. Then, from the previous theorem, $\gamma^*(x) \subset \langle P_1, P_2 \rangle + x$ and, since P_2 is a subgroup of $(R, +)$, it follows $\gamma^*(x) \subset \langle P_1 \rangle + P_2 + x$. Conversely, for all $z \in \langle P_1 \rangle + P_2 + x$ there exists $p_2 \in P_2$, $n \in \mathbb{N}$ such that $z \in x + p_2 + n(P_1 - P_1)$. Moreover, $p_2 \in P_2 = 1P_2^*1$ and, so,

$$\{z, x\} \subset xP_1^*(1P_2^*1)P_1^*(-P_1)P_1^* \dots P_1^*(-P_1)$$

that is $P_1^*(-P_1)$ n -times. Consequently, $\gamma^*(x) = \langle P_1 \rangle + P_2 + x$.

COROLLARY 3.4

If $(R, +, \cdot)$ is a ring and P_2 is a right ideal, then for all multiplicative P - H_V -rings over R , we have

$$R/\gamma^* = R/P_2.$$

Proof

Suppose that $\langle R, P_1^*, P_2^* \rangle$ is a multiplicative P - H_V -ring. Then, $\text{card} P_1 = 1$ and since the necessary and sufficient condition for the weak distributivity is $0 \in P_1$ [6], we have $P_1 = \{0\}$. Moreover, from the previous theorem, it follows $R/\gamma^* = R/P_2$.

THEOREM 3.5

Let $(R, +, \cdot)$ be a ring with unity and $P_1, P_2 \subset R$, $0 \in P_1$. If P_2 is a right ideal, then $\langle R, P_1^*, P_2^* \rangle$ is a H_V -field.

Proof

Since, for all $x \in R$, $0 \in xP_2P_2 \cap P_2xP_2$, then, from Theorem 2.4, it follows that $\langle R, P_1^*, P_2^* \rangle$ is a H_V -ring. On the other hand, the multiplication in the fundamental ring R/γ^* is defined, for all $x, y \in R$, in a usual manner

$$\gamma^*(x) \circ \gamma^*(y) = \{\gamma^*(z) : z \in \gamma^*(x)P_2^*\gamma^*(y)\}.$$

Moreover, $\gamma^*(x)P_2^*\gamma^*(y) \subset P_2$ and since $\gamma^*(z) = \langle P_1^* \rangle + P_2 + z$

(Theorem 3.3), it follows $\gamma^*(x) \circ \gamma^*(y) = \{\gamma^*(0)\}$.

Therefore, R/γ^* is a ring with a zero multiplication, so, $\langle R, P_1^*, P_2^* \rangle$ is a H_V -field.

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ON THE VERY THIN HYPERSTRUCTURES

Cornelia Gutan

Department of Mathematics

A.I. Cuza University

6600 Iasi, Romania

gutan@ucfma.univ-bpclermont.fr

Abstract

In this paper, the very thin hypergroupoids which verify some supplementary condition like (weak) associativity and reproductibility are studied. Some general constructions and structure theorems for this kind of hypergroupoids are given. Conditions under which a H_v -structure is a H_b -structure are also determined.

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1. Introduction

This paper deals with some very thin hyperstructures : these are structures on a non empty set in which only one *exceptional* couple of elements has not as product a singleton. The very thin hyperstructures, introduced by T. Vougiouklis in 1991 (see [V1]) for the representations of hypergroups by hypermatrices and generalized permutations, have direct applications in different branches of mathematics like combinatorics, group theory, probability and statistic, cryptography, computer science . These hyperstructures are all the more so interesting because they are very close by the classical univalent structures.

Many of the results presented in the following have been obtained in [G1] (see also [G2], [G3] , [G4]). These results are in a strong connexion with the classical theory of semigroups.

Results on the very thin hyperstructures have also been obtained by L. Konguetsof, S. Spartalis and T. Vougiouklis (see [KSV] and [V3]).

We designate by (H, a, b, A) a very thin hypergroupoid for which (a, b) is the exceptional couple. So the product $ab = A$ contains at least two elements and the *ordinary* couples have as product a singleton.

In any very thin hypergroupoid (H, a, b, A) we can associate to every $c \in A$ an underlying groupoid $(H, \cdot_c)$ for which the binary operation is defined by $x \cdot_c y = xy$, if the couple (x, y) is ordinary and $a \cdot_c b = c$.

Thus the hyperoperation of the very thin hypergroupoid (H, a, b, A) is the union of the operations of all its underlying groupoids $(H, \cdot_c)$, that is $xy = \bigcup_{c \in A} x \cdot_c y$. (When no confusion is possible we shall denote the operation on the underlying groupoids by “ $\cdot$ ” instead of “ $\cdot_c$ ”).

We also have an inverse construction. More precisely :

(I) Let $(H, \cdot)$ be a groupoid containing at least two elements and let us distinguish an exceptional couple (a, b) in H and a subset A of cardinal greater than or equal to two and such that $a \cdot b$ belongs to A . We define

on H a hyperoperation by $ab = A$ and $xy = x \cdot y$, if the couple (x, y) is ordinary. We obtain, of course, a very thin hypergroupoid (H, a, b, A) for which the initial given groupoid is an underlying one.

In fact, this inverse construction (I) means that we always obtain a very thin hypergroupoid if in a groupoid containing at least two elements the product of only one couple explodes such that it is contained between the components of the explosion.

Thus, the general problem is to establish the link between a very thin hypergroupoid and the family of its underlying groupoids in order to obtain the structure of this hypergroupoid and to realise a synthesis using classical univalent structures.

In this paper we study this problem for very thin hypergroupoids which also verify some supplementary conditions like (weak) associativity or reproductibility.

The study of the hyperstructures which admit classical underlying structures is also tackled in a monography of T. Vougiouklis ([V3]). He calls these hyperstructures H_b -structures, that is hyperstructures with basic univalent structures.

2. The very thin weakly associative hypergroupoids

It is obvious that if a (very thin) hypergroupoid is a H_b -semigroup then the hypergroupoid is weakly associative. So it is natural to ask if the converse is true, that is if every very thin weakly associative hypergroupoid H is a H_b -semigroup. In the following we answer to this question.

Let (H, a, b, A) be a very thin hypergroupoid and $G = H \setminus \{a\}$, $K = H \setminus \{b\}$.

We call *meioses* of a any decomposition $a = tu$, where t and u belong to G . Hence a has a meioses if and only if $GG \not\subseteq G$. In the same way, a *meioses* of b is any decomposition $b = vw$, where v and w belong to K . Therefore b has a meioses if and only if $KK \not\subseteq K$.

Hence generally, in order to answer to the previous question, there are

three situations to analyse :

- 1) There exist meiosis for both a and b .
- 2) There exist meiosis for b , but not for a . The situation when only a has meiosis results from a symmetrical treatment.
- 3) There is no meiosis for both a and b .

Corresponding to these three situations we have that :

- 1) Any very thin weakly associative hypergroupoid (H, a, b, A) such that there exist meiosis for both a and b is a H_b -semigroup and it contains an unique underlying semigroup.
- 2) If a very thin weakly associative hypergroupoid in which only one component of the exceptional couple has meiosis is a H_b -semigroup then it contains an unique underlying semigroup. Necessary and sufficient condition for the existence of such a semigroup are given in conditions $(*)$ and $(**)$ below (see page 6).
- 3) There exist very thin weakly associative hypergroupoids, for which the components of the exceptional couple have not meiosis, which are not H_b -semigroups. Also, there exist very thin weakly associative hypergroupoids for which the components of the exceptional couple have not meiosis which are H_b -semigroups and for which the underlying semigroup is not necessarily unique (see [G1] and [G2]).

We introduce the following notions. A triplet $(x, y, z) \in H^3$ is *ordinary* if each of the couples $(x, y), (xy, z), (y, z), (x, yz)$ is ordinary. Also a triplet (x, y, z) of an underlying groupoid $(H, \cdot)$ is *associative* if $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

We remark that :

- (R1) Every ordinary triplet of the very thin hypergroupoid H is associative in every underlying groupoid $(H, \cdot_c)$.
- (R2) The triplet (x, y, z) of H is ordinary in each of the following cases :
i) $x \neq a$ and $z \neq b$; ii) $x \neq a, y \neq a$ and $xy \neq a$; iii) $y \neq b, z \neq b$ and $yz \neq b$.

Hence (R1) and (R2) mean that the triplets (x, y, z) of $G \times H \times K$, of

$G \times G \times H$ if $xy \neq a$ and of $H \times K \times K$ if $yz \neq b$ are always ordinary.

1) We prove now that if both a and b have meiosis then for (H, a, b, A) there exists one and only one element $c \in A$ such that $(H, \cdot_c)$ is a semigroup.

Suppose that $tu = a$ is a meiosis of a and $vw = b$ is a meiosis of b . Using (R1) and (R2) we deduce that $t(ub) = t[u(vw)] = t[(uv)w] = [t(uv)]w = [(tu)v]w = (av)w$. Hence there exists at most one $c \in A$ such that in the underlying groupoid $(H, \cdot_c)$ we have $c = a \cdot b = t(ub) = (av)w$, for every meiosis $a = tu$ and $b = vw$.

We show now that for c chosen in this manner $(H, \cdot_c)$ is a semigroup. Consider x, y, z arbitrary elements of H .

i) If x, y are in G and $xy \neq a$ then, by (R2) ii), the triplet (x, y, z) is ordinary, so, according to (R1), it is also associative.

ii) Suppose that x, y are in G such that $xy = a$. If $z \neq b$ then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ because, by (R2) i), the triplet (x, y, z) is ordinary. If $z = b$ then $(x \cdot y) \cdot b = a \cdot b = x(yb) = x \cdot (y \cdot b)$.

Using i) and ii) we get that in the underlying groupoid $(H, \cdot_c)$ each triplet (x, y, z) of $G \times G \times H$ is associative. With a similar reasoning we can show that every triplet (x, y, z) of $H \times K \times K$ is associative.

iii) If $x \neq a$ and $y = a$ we have either $z \neq b$ and then $(x \cdot a) \cdot z = x \cdot (a \cdot z)$ because the triplet (x, a, z) is ordinary; or $z = b$ and then $(x \cdot a) \cdot b = (x \cdot a) \cdot (v \cdot w) = [(x \cdot a) \cdot v] \cdot w = [x \cdot (a \cdot v)] \cdot w = x \cdot [(a \cdot v) \cdot w] = x \cdot (a \cdot b)$.

Hence, we have proved that every triplet of $G \times H \times H$ is associative. In the same way, we obtain that every triplet of $H \times H \times K$ is associative.

iv) It remains to study the case $x = a$ and $z = b$. If $y \neq a$ then $(a \cdot y) \cdot b = [(t \cdot u) \cdot y] \cdot b = [t \cdot (u \cdot y)] \cdot b = t \cdot [(u \cdot y) \cdot b] = t \cdot [u \cdot (y \cdot b)] = (t \cdot u) \cdot (y \cdot b) = a \cdot (y \cdot b)$; and if $y = a$ then $(a \cdot a) \cdot b = [(t \cdot u) \cdot a] \cdot b = [t \cdot (u \cdot a)] \cdot b = t \cdot [(u \cdot a) \cdot b] = t \cdot [u \cdot (a \cdot b)] = (t \cdot u) \cdot (a \cdot b) = a \cdot (a \cdot b)$. Hence $(H, \cdot_c)$ is a semigroup.

2) Let (H, a, b, A) be a very thin weakly associative hypergroupoid such that only the component b of the exceptional couple (a, b) has meiosis.

In the following we establish that if $M = \{(v, w) \in K \times K \mid b = vw\}$, $L = \{v \in K \mid b \in vK\}$, $N = \{w \in K \mid b \in Kw\}$ then there exists $c \in A$ such that $(H, \cdot_c)$ is a semigroup if and only if the following two conditions are valid :

(*) The set $\{(av)w \mid (v, w) \in M\}$ is a singleton of which c is the unique element.

(**) If $M = L \times L (= N \times N)$ then $(a \cdot l) \cdot b = a \cdot (l \cdot b)$, for every $l \in L$.

It is obvious that the conditions (*) and (**) are necessary such that $(H, \cdot_c)$ be a semigroup. In the following we show that they are sufficient.

We prove first that if the condition (*) is satisfied then in the underlying groupoid $(H, \cdot_c)$ the triplets of $(G \times H \times H) \cup (H \times K \times K)$ are associative.

By (R1) and (R2) all the triplets of $G \times H \times K$ and $G \times G \times H$ are ordinary, hence associative. Also, every triplet $(a, y, z) \in \{a\} \times K \times K$ is associative because it is ordinary if $yz \neq b$ and if $yz = b$ then $(a \cdot y) \cdot z = c$, by hypothesis, and $a \cdot (y \cdot z) = a \cdot b = c$. Hence every triplet of $H \times K \times K$ is associative. Thus $(x \cdot a) \cdot b = (x \cdot a) \cdot (v \cdot w) = ((x \cdot a) \cdot v) \cdot w$ and as $((x \cdot a) \cdot v) \cdot w = (x \cdot (a \cdot v)) \cdot w = x \cdot ((a \cdot v) \cdot w) = x \cdot c = x \cdot (a \cdot b)$, we deduce that each triplet (x, a, b) of $G \times \{a\} \times \{b\}$ is associative. It follows that if the condition (*) is valid then in $(H, \cdot_c)$ the triplets of $(G \times H \times H) \cup (H \times K \times K)$ are associative.

Therefore $(H, \cdot_c)$ is a semigroup if and only if the triplets of $(\{a\} \times K \times \{b\}) \cup (\{a\} \times \{b\} \times H)$ are associative. So we examine these triplets (a, y, b) and (a, b, z) , where y, z are in K .

If $y \in K$ and $yL \neq b$ there exists a meiosis $vw = b$ such that $yv \neq b$. Therefore $(a \cdot y) \cdot b = (a \cdot y) \cdot (v \cdot w) = ((a \cdot y) \cdot v) \cdot w = (a \cdot (y \cdot v)) \cdot w = a \cdot ((y \cdot v) \cdot w) = a \cdot (y \cdot (v \cdot w)) = a \cdot (y \cdot b)$.

If $z \in K$ and $Nz \neq b$ we consider a meiosis $vw = b$ such that $wz \neq b$. Then $(a \cdot b) \cdot z = (a \cdot (v \cdot w)) \cdot z = ((a \cdot v) \cdot w) \cdot z = (a \cdot v) \cdot (w \cdot z) = a \cdot (v \cdot (w \cdot z)) = a \cdot ((v \cdot w) \cdot z) = a \cdot (b \cdot z)$.

If $y \in K$ and $yL = b$ then, for every meioses $(v, w) \in M$, we have $(a \cdot y) \cdot b = (a \cdot y) \cdot (v \cdot w) = (a \cdot b) \cdot w = (a \cdot b) \cdot N$ and $a \cdot (y \cdot b) = a \cdot (y \cdot (v \cdot w)) = a \cdot (b \cdot w) = a \cdot (b \cdot N)$.

If $z \in K$ and $Nz = b$ then, for every meioses $(v, w) \in M$, we have $(a \cdot b) \cdot z = (a \cdot (v \cdot w)) \cdot z = (a \cdot v) \cdot (w \cdot z) = (a \cdot v) \cdot b = (a \cdot L) \cdot b$ and $a \cdot (b \cdot z) = a \cdot ((v \cdot w) \cdot z) = a \cdot (v \cdot (w \cdot z)) = a \cdot (v \cdot b) = a \cdot (L \cdot b)$.

Thus, if there exists $w \in N$ such that $Nw \neq b$, we obtain that $(a \cdot y) \cdot b = (a \cdot b) \cdot w = a \cdot (b \cdot w) = a \cdot (y \cdot b)$, for every $y \in K$ such that $y \cdot L = b$. Hence all the triplets (a, y, b) , where $y \in K$, and all the triplets (a, b, z) , where $z \in K$, are associative.

Also, if there exists $y \in L$ such that $yL \neq b$, then all the triplets (a, b, z) , with $z \in K$, and all the triplets (a, y, b) , where $y \in K$, are associative.

Suppose now that the condition $(*)$ is satisfied and that $LL \neq \{b\}$ or $NN \neq \{b\}$. We prove that $(H, \cdot_c)$ is a semigroup. It remains to establish that the triplet (a, b, b) is associative.

Consider $(v, w) \in M$. Then $(a \cdot b) \cdot b = (a \cdot (v \cdot w)) \cdot b = ((a \cdot v) \cdot w) \cdot b = (a \cdot v) \cdot (w \cdot b) = a \cdot (v \cdot (w \cdot b)) = a \cdot ((v \cdot w) \cdot b) = a \cdot (b \cdot b)$. Hence $(H, \cdot_c)$ is a semigroup.

It remains to analyse the situation $LL = \{b\} = NN$. We have $L = N$ and $M = L \times L$. In this case $(a \cdot b) \cdot L = (a \cdot L) \cdot b$ is a singleton d , and $a \cdot (b \cdot L) = a \cdot (L \cdot b)$ is a singleton e . We may now conclude because, by condition $(*)$, $d = e$, therefore $(H, \cdot_c)$ is a semigroup.

Every very thin weakly asociative hypergroupoid for which the two components of the exceptional couple have meiosis can be obtained by the inverse construction (I). More precisely, in this case the inverse construction (I) must be applied to a semigroup S , with $\text{Card } S \geq 2$, in which we distinguish two elements a, b such that $S \setminus \{a\}$ and $S \setminus \{b\}$ are not subsemigroups of S . We obtain that (S, a, b, A) is a very thin weakly associative

hypergroupoid in which the components a, b of the exceptional couple (a, b) have both meiosis.

In order to obtain very thin weakly associative hypergroupoids in which only one component of the exceptional couple has meiosis we use once again the inverse construction (I). This must be applied to a semigroup S , with $\text{Card } S \geq 2$, in which we distinguish two distinct elements a, b such that $S \setminus \{a\}$ is a subsemigroup of S , but $S \setminus \{b\}$ is not a subsemigroup of S . We obtain that (S, a, b, A) is a very thin weakly associative hypergroupoid in which only the component b of the exceptional couple (a, b) has meiosis.

We show now that there exist very thin weakly associative hypergroupoids (H, a, b, A) for which $GG \subset G$ and $KK \not\subset K$ and such that they do not verify the condition $(*)$ (hence they are not H_b -semigroups).

Example. Let H be a set such that $\text{Card } H \geq 3$ and a, b be two distinct elements of H . We define a hyperoperation on H by $ab = H, a(H \setminus \{a, b\}) = a$ and all the other products are equals to b . Then (H, a, b, A) is a very thin weakly associative hypergroupoid for which $M = K^2 \setminus (\{a\} \times (H \setminus \{a, b\}))$ and $L = N = K$. Then, if $v \in H \setminus \{a, b\}$, we have $(v, v) \in M, (v, a) \in M$ and $(av)v = av = a, (av)a = aa = b$. Hence the condition $(*)$ is not verified.

The following examples correspond to the other possible situations.

Example. Let H be a set such that $\text{Card } H \geq 3$. Consider a, b two distinct elements of H and define on H an operation by $x \cdot a = a$ and $x \cdot (H \setminus \{a\}) = b$, for all x in H . If we define $ab = H$ and preserve the operation in the rest of H then H endowed with this hyperoperation is a very thin weakly associative hypergroupoid (H, a, b, H) for which $G = H \setminus \{a\}, GG = \{b\} \subset G, K = H \setminus \{b\}, KK = \{a, b\} \not\subset K, M = (H \setminus \{b\}) \times (H \setminus \{a, b\}), L = K$. The hypergroupoid (H, a, b, H) satisfies the condition $(*)$ and $LL = \{a, b\} \neq \{b\}$.

Exemple. Let a, b be two distinct elements of a semigroup $(H, \cdot)$, where $\text{Card } H \geq 3$ and $a \cdot y = a, x \cdot y = b$, for all x, y in H , with $x \neq a$. We define on H a structure of very thin weakly associative hypergroupoid by $ab = H$ and $xy = x \cdot y$, for all x and y in H such that $(x, y) \neq (a, b)$. In this hypergroupoid $M = (H \setminus \{a, b\}) \times (H \setminus \{b\}) = (K \setminus \{a\}) \times K$, $L = H \setminus \{a, b\}$, $N = H \setminus \{b\}$, $LL = \{b\}$, $NN = \{a, b\} \neq \{b\}$. Hence the conditions $(*)$ and $NN \neq \{b\}$ are satisfied.

Exemple. Let T be a null semigroup with a zero element b . Hence $x \cdot y = b$, for all x, y in T . Consider U the ideal extension of an arbitrary semigroup I by T (i.e. U is a semigroup such that $U = I \cup T$, where $I \cap T = \emptyset$, I is an ideal of U and the Rees quotient U/I is isomorphic to T). Let S be the semigroup obtained from U by adjoining a zero 0 . Consider $H = S$ and the hypergroupoid $(H, 0, b, H)$. This is a very thin weakly associative hypergroupoid for which $G = U$, $K = S \setminus \{b\}$, $M = (T \setminus \{b\}) \times (T \setminus \{b\})$, $LL = NN = \{b\}$, $d = (0 \cdot L) \cdot b = 0$, $e = 0 \cdot (L \cdot b) = 0$. Hence the conditions $(*)$ et $(**)$ are valid.

We give now an example where $(*)$ is verified but $d \neq e$.

Exemple. Let H be a set such that $\text{Card } H \geq 5$. Consider a, b, x, y, z distinct elements of H and define on H a hyperoperation by $ab = H, aa = b, ba = x, xa = bb = y$ and all the other products are equal to z . Then (H, a, b, H) is a very thin weakly associative hypergroupoid and $G = H \setminus \{a\}$, $GG = \{y, z\} \subset G$, $K = H \setminus \{b\}$, $b \in KK$, $M = \{(a, a)\}$, whence $(*)$ is verified ($\{(av)w \mid (v, w) \in M\} = \{(aa)a\} = \{x\}$), $L = N = \{a\}$, $LL = NN = \{b\}$. We also have $d = (a \cdot b) \cdot L = xa = y$ and $e = a \cdot (b \cdot L) = a(ba) = z$. Hence $d \neq e$.

3. The very thin quasihypergroups

Using the inverse construction (I) in which we start from a quasi-groupoid containing at least two elements, we obtain very thin quasihypergroups which are also H_b -quasigroups.

Another remarkable method for obtaining very thin quasihypergroups is the following :

(M) Let Q be a quasigroup and add to Q an element a . Consider $Q \cup \{a\}$ and $A \in \{Q, Q \cup \{a\}\}$, where $A \neq Q$ if Q is a singleton. We define a hyperoperation by : $aa = A, aQ = Qa = a$ and preserving the initial operation on Q in the rest. In this way, $Q \cup \{a\}$ endowed with this hyperoperation is a very thin quasihypergroup $(Q \cup \{a\}, a, a, A)$.

All the very thin quasihypergroups of cardinal greater than or equal to three and such that at least one of the components of the exceptional couple has not meiosis can be obtained by this method but there are quasihypergroups which cannot be obtained by the method (M) or by the inverse construction (I) (see [G1], [G2]).

4. The very thin hypergroups

It turns out that method (M) and the inverse construction (I) are sufficient to obtain all the very thin quasihypergroups which also verify supplementary conditions like associativity or weak associativity. Indeed, every very thin hypergroup can be obtained by the previous method (M) starting from a group (see [G3]).

A similar result for the case of finite very thin hypergroup has been obtained by T. Vougiouklis ([V1]).

5. The very thin weakly associative quasihypergroups

The very thin weakly associative quasihypergroups are also called H_v -groups. They can all be obtained by the previous method (M) and by the inverse construction (I) starting from a group. Hence, every very thin H_v -group is either a H_b -group or a very thin hypergroup (see [G3]).

6. The very thin semihypergroups

For a very thin semihypergroup (H, a, b, A) the subsets G and K are subsemigroups in every of its underlying groupoids (see [G3]). Whence

there is no meioses for the components a and b of the exceptional couple. Also, if $a \neq b$, then $H \setminus \{a, b\}$ must be a subsemigroup of both G and K . In consequence, we obtain that the characterization for very thin semihypergroups can be reduced to some problems on semigroups (for instance, one of these problems is to find all the embeddings of a semigroup in a semigroup having only one element in plus).

If $a = b$ (this is a necessary condition in the case of the very thin commutative semihypergroups) we consider the semigroup $S = G$ and its subsets $T = \{s \in S \mid as = a\}$ and $U = S \setminus T$. This divides S in two disjointed subsets. We obtain that : i) if T is not empty then it is a subsemigroup of S ; ii) if U is not empty then it is a prime ideal of S or it is S itself ; iii) the application $\psi : U \longrightarrow U, \psi(u) = au$, for every $u \in U$, is a translation of U .

Using these remarks we get the structure of the very thin commutative semihypergroups using classical notions of semigroup theory like kernel, translation, prime ideal. We start with a commutative semigroup S and an element a which does not belong to S . Consider $H = S \cup \{a\}$, $\varphi : S \longrightarrow H$ an application and A a subset of H containing at least two elements. We define on H a hyperoperation by : $aa = A, xS = Sx = \varphi(x)$, for every $x \in S$, and preserving the operation on the semigroup S in the rest.

Let $H(S, a, \varphi, A)$ be the very thin hypergroupoid obtained in this manner. It follows that for every very thin commutative semihypergroup H there exist S, a, φ, A such that $H = H(S, a, \varphi, A)$. In the following we give necessary and sufficient conditions which must be verify by S, φ and A such that $H(S, a, \varphi, A)$ be a very thin commutative semihypergroup.

Denote $T = \varphi^{-1}(a)$, $U = S \setminus T$, $B = A \cap S, C = B \cap T$ and $D = B \cap U$. Consider $\psi = \varphi|_U : U \longrightarrow U$ the restriction of φ on U and the subset of S defined by $E(f) = \{s \in S \mid \lambda_s = f\}$, where $f : U \longrightarrow U$ is an application and λ_s is the inner translation associated to every $s \in S$, that is $\lambda_s : U \longrightarrow U, \lambda_s(u) = su$, for every $u \in U$.

We obtain that $H(S, a, \varphi, A)$ is a very thin commutative semihypergroup if and only if

- i) U is a prime ideal of S , or U is the empty set or $U = S$;
- ii) $\psi : U \longrightarrow U$ is an application ;
- iii) B a subset of S

all these satisfying the following conditions :

$$\left\{ \begin{array}{l} \psi \text{ is a translation of } I \text{ and } \psi = \lambda_t \circ \psi = \psi \circ \lambda_t, \text{ for every } t \in T \\ B \subset E(\psi^2), B \neq \emptyset \text{ and if } a \notin A \text{ then Card } B \geq 2 \\ \psi^2 = \psi \quad \text{if } a \in A; \\ C = K(T) \text{ or } \emptyset, D = D \cdot t, \text{ for every } t \in T. \end{array} \right.$$

The possible situations are presented in the following table :

		I.	II.
		$\langle a \in A \rangle$	$\langle a \notin A \rangle$
$U = \emptyset$		$B = K(S)$	$B = K(S), \text{ Card } K(S) \geq 2$
$U = S$		$\psi = \lambda_\alpha, \alpha \in \text{Idemp}(U)$ $\emptyset \neq B \subset E(\lambda_\alpha)$	ψ a translation of U $B \subset E(\psi^2), \text{ Card } B \geq 2$
$\emptyset \neq U \neq S$	$D = \emptyset$	$\psi = \lambda_\alpha, \alpha \in B$ $B = C = K(T)$ $B \subset E(\lambda_{\alpha^2})$	ψ a translation of U $B = C = K(T), \text{ Card } B \geq 2$ $B \subset E(\psi^2)$ $\psi = \psi \lambda_t = \lambda_t \psi, \text{ for every } t \in T$
	$C \neq \emptyset$ $D \neq \emptyset$	$\psi = \lambda_\alpha, \alpha \in \text{Idemp}(U)$ $C = K(T), D = \{\alpha\}$ $C \subset E(\lambda_\alpha)$ $\alpha T = \alpha$	$\psi = \lambda_\beta, \beta \in I$ $C = K(T), D = \{\beta^2\}$ $C \subset E(\lambda_{\beta^2})$ $\beta u = \beta u T, \text{ for every } u \in I$
	$C = \emptyset$	$\psi = \lambda_\alpha, \alpha \in U$ $\emptyset \neq B \subset E(\lambda_\alpha) \cap U$ $\lambda_{\alpha^2} = \lambda_\alpha$ $Bt = B, \text{ for every } t \in T$	ψ a translation of U $B \subset E(\psi^2) \cap U, \text{ Card } B \geq 2$ $\psi = \psi \lambda_t = \lambda_t \psi, \text{ for every } t \in T$ $Bt = B, \text{ for every } t \in T$

Hence we obtain two classes corresponding to the situations $a \in A$ or $a \notin A$. In each of these two classes there are three genera depending on the fact that U is empty, or equal to S or a prime ideal of S ; the least genus is composed by three species depending on the fact that D is empty or equal to B or $\emptyset \neq D \neq B$ (see [G4]).

Using these results we obtain the structure of the very thin commutative idempotent semihypergroups.

There are three kinds of such semigroups. More precisely, let $(S, \cdot)$ be a commutative idempotent semigroup. Hence $(S, \cdot)$ is a semilattice. We can define on S an order relation by $x \leq y$ if and only if $x \cdot y = y$, for all x, y in S . Consider an element a which does not belong to S and T a subset of S . We obtain that $H = S \cup \{a\}$ is a very thin commutative idempotent semihypergroup if and only if the hyperoperation on H is defined in one of the following three manners :

- Let $\alpha \in S$ and $T \leq \alpha$. Then $aa = \{a, \alpha\}$, $Ta = aT = a$, $ax = xa = \alpha \cdot x$, for all $x \in I$, and $xy = x \cdot y$, for all x, y in S .
- Let $\alpha \in S$ and $T < \alpha$ such that $x \cdot y \in T$ if and only if both x and y belong to T . Then $aa = \{a, \alpha\}$, $Ta = aT = a$, $ax = xa = \alpha \cdot x$, for all $x \in I$, and $xy = x \cdot y$, for all x, y in S .
- Let $\alpha \in S$, $\beta \in S$, $\alpha < \beta$, $[\alpha, \beta] = \{\alpha, \beta\}$ and $T \leq \alpha$. Then $aa = \{a, \alpha, \beta\}$, $Ta = aT = a$, $ax = xa = \beta \cdot x$, for all $x \in I$, and $xy = x \cdot y$, for all x, y in S .

As an application, we get the structure of the very thin hyperlattices (see [G5]).

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School of Mathematical Science, Fudan University
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Sankar Hajra

Indian Physical Society
2A & 2B Raja S. C. Mullick Road
Kolkata 700032 India

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The Citadel
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Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

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Department of Mathematics, School of Engineering
Democritus University of Thrace, 67100 Xanthi, Greece

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Cornelia Gutan

Department of Mathematics
Al.I. Cuza University
6600 Iasi, Romania